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Test Four

This is a self-diagnostic test. Every pair of questions relates to a worksheet in a series available in the MUMS the WORD series. For example question 5 relates to worksheet 4.5 *Binomial Coefficients*. If you score 100% on this test and test 3 then we feel you are adequately prepared for your first year mathematics course. For those of you who had trouble with a few of the questions, we recommend working through the appproriate worksheets and associated computer aided learning packages in this series.

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1. (a) Differentiate $y = \log(3x + 2)$

   (b) Find $\frac{dy}{dx}$ if $y = x^2 \cos x$

2. (a) Given the following monotonically increasing function, find an upper and lower limit for the area under the curve between 0 and 4.

x	0	1	2	3	4
$g(x)$	2	3	5	6.5	7

   (b) Find the area under the curve $y = x^2 + 1$ between $x = 1$ and $x = 3$.

3. Evaluate the following indefinite integrals:

   (a) $\displaystyle\int \frac{1}{x}\, dx$

   (b) $\displaystyle\int \sec^2 x\, dx$

4. (a) Given $\dfrac{d^2x}{dt^2} = 9$ for all x and when $t = 0$ we have $\frac{dx}{dt} = 4$ and $x = 3$. What is x as a function of t?

   (b) A population $P(t)$ is given by the following formula:

   $$P(t) = P(0)e^{kt}$$

   If the initial population is 1000, and the growth rate is 0.01, what is the population at $t = 100$? (You can leave the answer in terms of the natural exponential)

5. Divide $6x^3 + x^2 - x + 4$ by $x + 1$.

6. (a) What is the coefficient of x^2 in the expansion of $(5x - 1)^5$?

   (b) Evaluate $\dfrac{6!}{4!2!}$.

Worksheet 4.1 More Differentiation

Section 1 The chain rule

In the last worksheet, you were shown how to find the derivative of functions like $e^{f(x)}$ and $\sin g(x)$. This section gives a method of differentiating those functions which are what we call composite functions. The method is called the chain rule. The chain rule allows us to differentiate composite functions. Composite functions are functions of functions, and can be written as

$$g(x) = f(u(x))$$

So if $u(x) = x^2$ and $f(u) = \cos u$, then

$$f(u(x)) = \cos x^2$$

The derivative of such functions is given by the following rule:

$$g'(x) = \frac{du(x)}{dx} \times \frac{df}{du}$$

So for our example of $g(x) = f(u(x)) = \cos x^2$ we have

$$\frac{df}{du} = -\sin u = -\sin x^2 \quad \text{and} \quad \frac{dg}{dx} = (2x) \times (-\sin x^2)$$

The trick is working out which function is the f and which is the u – it is what you do to the input first.

Example 1 : Differentiate e^{5x^2}. Let $u(x) = 5x^2$ and $f(u) = e^u$. If $g(x) = f(u(x))$ then

$$g'(x) = u'(x) \times \frac{df}{du}$$

We have $\frac{du}{dx} = 10x$ and $\frac{df}{du} = e^u = e^{5x^2}$ so that

$$g'(x) = 10xe^{5x^2}$$

Example 2 : Differentiate $g(x) = \sin(e^x)$. We let $u(x) = e^x$ and $f(u) = \sin u$. Then

$$\begin{aligned} u'(x) &= e^x \\ \frac{df}{du} &= \cos u = \cos e^x \\ \frac{dg}{dx} &= u'(x) \times \frac{df}{du} \\ &= e^x \cos(e^x) \end{aligned}$$

Example 3 : Differentiate $y = (6x^2 + 3)^4$. We let $u(x) = 6x^2 + 3$ and $f(u) = u^4$. Then

$$\begin{aligned} u'(x) &= 12x \\ \frac{df}{du} &= 4u^3 = 4(6x^2+3)^3 \\ \frac{dy}{dx} &= u'(x) \times f'(u) \\ &= 12x \times 4(6x^2+3)^3 \end{aligned}$$

Example 4 : Differentiate $y = (3x + 2)^4$. Let $u(x) = 3x + 2$. Then

$$\frac{dy}{dx} = 3 \times 4 \times (3x+2)^3 = 12(3x+2)^3.$$

Exercises:

1. Differentiate the following with respect to x.

(a) $\sin 3x$
(b) $\tan(-2x)$
(c) $\cos 6x^2$
(d) $(4x + 5)^5$
(e) $(6x - 1)^3$
(f) $(3x^2 + 1)^4$
(g) e^{4x}
(h) $7e^{2x}$
(i) $e^{\sin x}$
(j) $e^{\cos x}$
(k) $(6 - 2x)^3$
(l) $(7 - x)^4$

Section 2 The product rule

The product rule gives us a method of working out the derivative of a function which can be written as the product of functions. Examples of such functions are $x^2 \sin x$, $5x \log x$, and $e^x \cos x$. These functions all have the general form

$$\begin{aligned} h(x) &= f(x)g(x) \quad \text{or in simpler terms} \\ h &= fg \end{aligned}$$

For functions that are written in this form, the product rule says:

$$\frac{dh}{dx} = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx} \quad \text{or} \quad h' = f'g + fg'$$

When first working with the product rule, it is wise to write down all the steps in the calculation to avoid any confusion.

Example 1 : Differentiate $h(x) = x^2 \sin x$.
Let $f(x) = x^2$ and $g(x) = \sin x$. Then $f'(x) = 2x$ and $g'(x) = \cos x$, which gives

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 2x \sin x + x^2 \cos x \end{aligned}$$

Note that in terms such as $\cos x \times x^2$, it is less ambiguous to write $x^2 \cos x$ to make it clear that we are not taking the cos of the x^2 term.

Example 2 : Differentiate $h(x) = 5x \log x$.
Let $f(x) = 5x$ and $g(x) = \log x$ so that $f'(x) = 5$ and $g'(x) = \frac{1}{x}$. Then

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 5 \log x + 5x \times \frac{1}{x} \\ &= 5 \log x + 5 \end{aligned}$$

Example 3 : Differentiate $p(x) = e^x \cos x$.
Let $a(x) = e^x$ and $b(x) = \cos x$. Then $a'(x) = e^x$ and $b'(x) = -\sin x$, so that

$$\begin{aligned} p'(x) &= a'(x)b(x) + a(x)b'(x) \\ &= e^x \cos x + e^x(-\sin x) \\ &= e^x(\cos x - \sin x) \end{aligned}$$

Example 4 : Differentiate $h(x) = 3x^2 e^x$.
Let $f(x) = 3x^2$ and $g(x) = e^x$ so that $f'(x) = 6x$ and $g'(x) = e^x$. Then

$$\begin{aligned} h'(x) &= 6xe^x + 3x^2 e^x \\ &= 3x(2e^x + xe^x) \\ &= 3x(x+2)e^x \end{aligned}$$

Note that, when using the product rule, it makes no difference which part of the whole function we call $f(x)$ or $g(x)$ (so long as we are able to differentiate the f or g that we choose). So in example 4, we could have let $f(x) = 6e^x$ and $g(x) = x$ and the final result for $h'(x)$ would have been the same.

Exercises:

1. Differentiate the following with respect to x.

(a) $x^2 \sin x$
(b) $4xe^{3x}$
(c) $x^2 e^{3x}$
(d) $x \cos x$
(e) $4x \log(2x+1)$
(f) $x^2 \log(x+2)$
(g) $x^2 e^{x^3}$
(h) $(3x+1)(x+1)^3$
(i) $3x(x+2)^3$
(j) $(4x-1)e^{2x}$
(k) $\sin xe^{2x}$
(l) $\cos(2x)e^{4x}$

Section 3 THE QUOTIENT RULE

The quotient rule is the last rule for differentiation that will be discussed in these worksheets. The quotient rule is derived from the product rule and the chain rule; the derivation is given at the end of the worksheet for those that are interested. The quotient rule helps to differentiate functions like $\frac{e^{2x}}{x^2}$, $\frac{x^2}{\cos x}$ and $\frac{x^2+1}{x^3+3}$. The general form of such expressions is given by $k(x) = \frac{u(x)}{v(x)}$, and the quotient rule says that

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \quad \text{or}$$
$$k' = \frac{u'v - uv'}{v^2}$$

It is a good idea to do some bookkeeping when using the quotient rule.

Example 1 : Differentiate $k(x) = \frac{e^{2x}}{x^2}$.
Let $u(x) = e^{2x}$ and $v(x) = x^2$. Then $u'(x) = 2e^{2x}$ and $v'(x) = 2x$, which gives

$$\begin{aligned} k'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \\ &= \frac{2e^{2x}x^2 - e^{2x}2x}{(x^2)^2} \\ &= \frac{2xe^{2x}(x-1)}{x^4} \\ &= \frac{2e^{2x}(x-1)}{x^3} \end{aligned}$$

Note that the choice of $u(x)$ and $v(x)$ are not interchangeable as in the product rule. Given the complicated appearance of the quotient rule, it is wise to be consistent and always let $u(x)$ be the numerator and $v(x)$ the denominator.

Example 2 : Differentiate $p(x) = \frac{x^2}{\cos x}$. Let $u(x) = x^2$ and $v(x) = \cos x$. Then $u'(x) = 2x$ and $v'(x) = -\sin x$, so that

$$\begin{aligned} p'(x) &= \frac{2x\cos x - x^2(-\sin x)}{(\cos x)^2} \\ &= \frac{2x\cos x + x^2\sin x}{\cos^2 x} \\ &= \frac{x(2\cos x + x\sin x)}{\cos^2 x} \end{aligned}$$

Example 3 : Differentiate $p(x) = \frac{x^2+1}{x^3+3}$.
Let $u(x) = x^2 + 1$ and $v(x) = x^3 + 3$. Then $u'(x) = 2x$ and $v'(x) = 3x^2$, so that

$$\begin{aligned} p'(x) &= \frac{2x(x^3+3) - (x^2+1)3x^2}{(x^3+3)^2} \\ &= \frac{6x - 3x^2 - x^4}{(x^3+3)^2} \end{aligned}$$

We now derive the quotient rule from the product and chain rule; skip the derivation if you don't feel the need to know. Let

$$k(x) = \frac{u(x)}{v(x)} = u(x)(v(x))^{-1}$$

We now use the product rule and let $f(x) = u(x)$ and $g(x) = (v(x))^{-1}$. Then $f'(x) = u'(x)$ and the derivative of $g(x)$ is given by the chain rule:

$$\begin{aligned} g'(x) &= -1(v'(x))(v(x))^{-2} \\ &= \frac{-v'(x)}{(v(x))^2} \end{aligned}$$

Using the product rule on $k(x)$ (the thing we are trying to differentiate), we get

$$\begin{aligned} k'(x) &= u'(x)(v(x))^{-1} + u(x) \times \frac{-v'(x)}{(v(x))^2} \\ &= \frac{u'(x)}{v(x)} - \frac{u(x)v'(x)}{(v(x))^2} \\ &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \end{aligned}$$

This is the quotient rule.

Exercises:

1. Differentiate the following with respect to x.

(a) $\dfrac{x^3}{x^2+1}$

(b) $\dfrac{x^2+3}{x+1}$

(c) $\dfrac{x-1}{2x+3}$

(d) $\dfrac{e^{2x}}{x-3}$

(e) $\dfrac{\sin x}{x^2}$

(f) $\dfrac{\sin x}{\cos x}$

(g) $\dfrac{3x}{x^2-2}$

(h) $\dfrac{x+6}{x-4}$

(i) $\dfrac{6e^x}{x+5}$

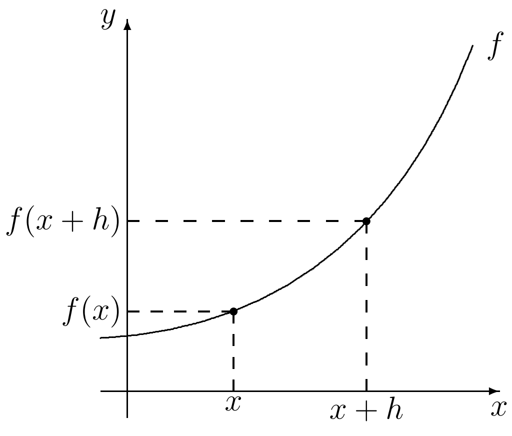
(j) $\dfrac{e^{2x}}{\sin x}$

Section 4 Equations of tangents and normals to curves

When the topic of differentiation was first introduced in section 1 of Worksheet 3.8, we said that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This was motivated from this picture:

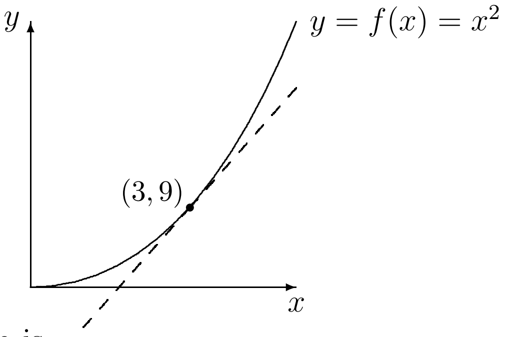


As $h \to 0$ the secant joining $(x, f(x))$ and $(x+h, f(x+h))$ becomes a better and better approximation to the tangent of f at the point $(x, f(x))$. We will now find the equation of this tangent. Recall that the tangent is just a straight line and it passes through the point $(x, f(x))$ on the curve. We have already found the equation of a straight line through a given point, say (x_1, y_1), with a given slope, say m – this was done in Worksheet 2.10. The equation of such a straight line is

$$y - y_1 = m(x - x_1).$$

Example 1 : Find the equation of the tangent to the curve $y = x^2$ at the point $(3, 9)$.

A piece of the function is drawn as well as the tangent.



The derivative of the function is

$$\frac{dy}{dx} = 2x.$$

At the point $(3, 9)$, $\dfrac{dy}{dx} = 2 \times 3 = 6$ so that the slope of the tangent line is 6. Now, a point that lies on the tangent line is $(3, 9)$, so the equation of the tangent line is

$$\begin{aligned} y - 9 &= 6(x - 3) \\ y - 9 &= 6x - 18 \\ y &= 6x - 9 \end{aligned}$$

The equation of the tangent of $y = x^2$ at $(3, 9)$ is $y = 6x - 9$.

Example 2 : Find the equation of the tangent to the curve $y = x^3 - x + 4$ at the point $(1, 4)$.

We will find the equation without drawing the graph. We have

$$\frac{dy}{dx} = 3x^2 - 1,$$

so the slope of the tangent at $x = 1$ is $3(1)^2 - 1 = 2$. A point that the tangent passes through is $(1, 4)$, so the equation must be given by

$$\begin{aligned} y - 4 &= 2(x - 1) \\ y - 4 &= 2x - 2 \\ y &= 2x + 2 \end{aligned}$$

The equation of the tangent of $y = x^3 - x + 4$ at $(1, 4)$ is $y = 2x + 2$.

Example 3 : Find the equation of the tangent to the curve $y = e^{2x}$ at the point $(3, e^6)$.

We have

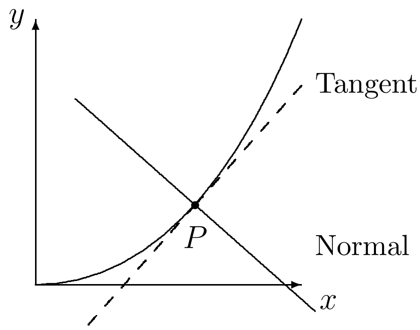
$$\frac{dy}{dx} = 2e^{2x},$$

so the slope of the tangent at $x = 3$ is $2e^6$. A point that the tangent passes through is $(3, e^6)$, so the equation must be given by

$$\begin{aligned} y - e^6 &= 2e^6(x-3) \\ y - e^6 &= 2e^6x - 6e^6 \\ y &= 2e^6x - 5e^6 \\ y &= e^6(2x-5) \end{aligned}$$

The equation of the tangent of $y = e^{2x}$ at $(3, e^6)$ is $y = e^6(2x - 5)$.

The *normal* to a curve at a particular point is the straight line that passes through the point in question on the curve and is perpendicular to the tangent to the curve.



Example 4 : Find the equation of the normal to the curve $y = x^2$ at the point $(3, 9)$.

From example 1, the slope of the tangent is 6, so the gradient of the normal to the tangent is $-\frac{1}{6}$. (Recall that in section 3 of Worksheet 2.10 we said that if two lines are perpendicular, then the product of their slopes is -1.) So the equation of the normal at the point $(3, 9)$ is

$$\begin{aligned} y - 9 &= -\frac{1}{6}(x-3) \\ y - 9 &= -\frac{1}{6}x + \frac{1}{2} \\ y &= -\frac{1}{6}x + \frac{19}{2} \end{aligned}$$

Example 5 : Find the equations of the tangent and normal to the curve $y = x^3 - 5x + 6$ at $(-3, -6)$.

We have $\dfrac{dy}{dx} = 3x^2 - 5$, so the slope of the tangent when $x = -3$ is 22. The equation of the tangent is then given by

$$\begin{aligned} y - (-6) &= 22(x - (-3)) \\ y &= 22x + 60 \end{aligned}$$

The equation of the normal is given by

$$\begin{aligned} y - (-6) &= -\frac{1}{22}(x - (-3)) \\ y &= -\frac{1}{22}x - \frac{135}{22} \end{aligned}$$

Exercises:

1. Find the equation of the tangent to the curve
   (a) $y = x^2 - 4x + 6$ at the point $(-2, 18)$
   (b) $y = 6 - x^2$ when $x = 3$
   (c) $y = x^3 - 4x + 30$ when $x = -5$
2. Find the equation normal to the curve
   (a) $y = 8 - 3x^2$ at the point $(4, -40)$
   (b) $y = x^3 - 2x^2 + 6$ when $x = -1$
   (c) $y = 6/x$ at the point $(-2, -3)$.
3. Find the equation of the tangent to the curve $y = 3x^2 - 2x + 4$ at the point $(1, 5)$ and also find the point where the tangent cuts the x axis.

Exercises for Worksheet 4.1

1. Differentiate the following

   (a) $y = \frac{1}{x^2} - 6x + 4$

   (b) $y = xe^{2x}$

   (c) $y = \sin 2x - \cos 4x$

   (d) $y = (2x+1)^3(x+2)$

   (e) $y = 4x \sin x$

   (f) $y = \log(x^2+1)$

   (g) $y = x \log x$

   (h) $y = (\sin x)^2$

   (i) $y = e^x \sin x$

   (j) $y = \frac{6x+1}{x-4}$

   (k) $y = \frac{x^2}{x+3}$

   (l) $y = \frac{e^x}{x-2}$

2. (a) Find the equation of the tangent to the curve $y = x^2 \log x$ at the point (e, e^2).

   (b) If $f(x) = \sin 2x - \cos 4x$, find $f'(\frac{\pi}{4})$.

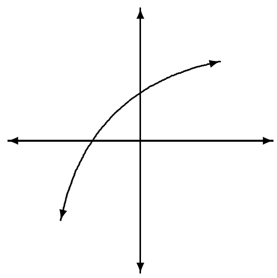
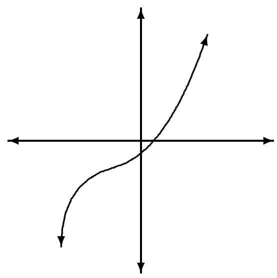
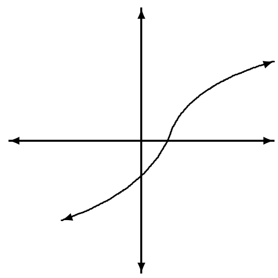
   (c) If $y = (x^2-1)(1+x)$, show that $x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2x - 2 = 0$.

   (d) Find the turning point of the curve $y = x^2 + 3x - 4$ and state whether it is a maximum or minimum turning point.

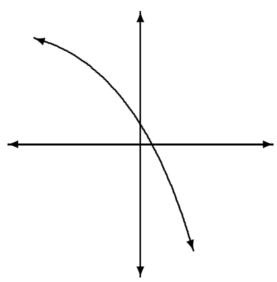
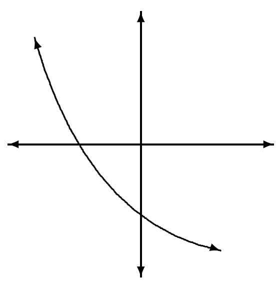
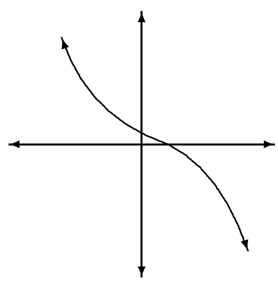
Worksheet 4.2 Introduction to Integration

Section 1 Left and Right Rectangles

A function f which has the property that if $b > a$ then $f(b) > f(a)$ is called monotonically increasing - as the input increases, then the output increases. The 'monotonically' part comes from the property that there are no maxima or minima. The slope of a monotonically increasing function will always be greater than or equal to zero, and it will only equal zero at a point of inflection. Here are some examples of the graphs of monotonically increasing functions:

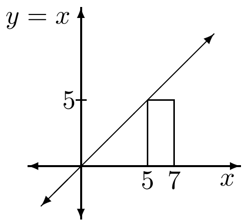




A monotonically decreasing function f is one which has the property that if $b > a$, then $f(b) < f(a)$. In other words, as the input gets larger, the output gets smaller. The slope of a monotonically decreasing function is always less than or equal to zero, and is only zero at a point of inflection. Here are some examples of the graphs of monotonically decreasing functions:

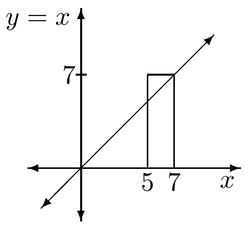
Sometimes it is important for us to be able to estimate the area under a curve, which might represent a quantity in which we are interested. For example, if we had a graph of a motorist's velocity as a function of time for a journey that lasted an hour, the area under the curve would represent the distance travelled over the journey. We now give a method that can be used to estimate the area under certain types of curves, namely those that are either monotonically increasing or decreasing. Note that all functions can be broken up into a sequence of parts, each of which is either (a) monotonically increasing or (b) monotonically decreasing or (c) horizontal. A vertical line is not a function as it does not have the property that each input value has one and only one output value. The estimation method involves splitting the area up into rectangles to give a lower and upper bound to the area under the curve.

Take the function $y = x$ as an example, and say we wish to know the area under the curve $y = x$ between $x = 5$ and $x = 7$. First we draw the graph:



If we draw a rectangle the height of the value of the function at $x = 5$ which stretches across to the same height above $x = 7$ we get the shaded region. This is called a left rectangle, as its height is given by the function value on the left hand side of the interval. Since $f(5) = 5$, the height of the rectangle is 5, and the width is $7 - 5 = 2$, so the area of the rectangle is $5 \times 2 = 10$.

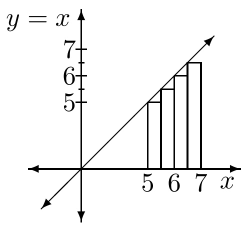
Now we draw the following diagram:

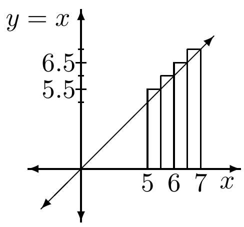


For the height of the rectangle, we use the value of the function at $x = 7$, which is $f(7) = 7$. This is called a right rectangle, and it has area $7 \times 2 = 14$. The area under the line $y = x$ between $x = 5$ and $x = 7$ must lie somewhere between 10 and 14 since the area of the right rectangle is bigger than the area under the line, and the area of the left rectangle is smaller than than the area under the line. Thus, if A is the area then,

$$10 < A < 14$$

We can get a closer approximation to the area under this line by breaking the interval into smaller pieces. Say we look at the function $y = x$ at every 1/2 unit, and add up the area of the rectangles formed by using as our intervals: [5,5.5],[5.5,6],[6,6.5], and [6.5,7]. The left rectangles are shown in figure 3, and the right rectangles in figure 4.





Note : The area of each rectangle is found by multiplying the base by the height. The area given by the left rectangles is the sum:

$$f(5)(5.5-5)+f(5.5)(6-5.5)+f(6)(6.5-6)+f(6.5)(7-6.5)=\frac{5}{2}+\frac{11}{4}+\frac{6}{2}+\frac{13}{4}=11\frac{1}{2}$$

The area given by the right rectangles is

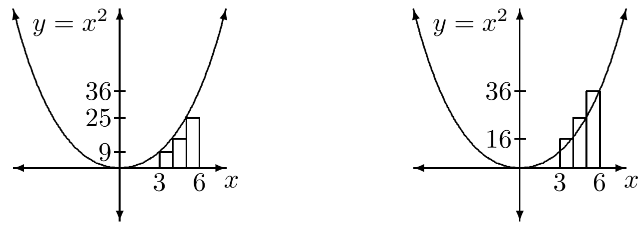
$$f(5.5)\frac{1}{2}+f(6)\frac{1}{2}+f(6.5)\frac{1}{2}+f(7)\frac{1}{2}=\frac{11}{4}+\frac{6}{2}+\frac{13}{4}+\frac{7}{2}=12\frac{1}{2}$$

The lower and upper bounds on the area A are now given by:

$$11\frac{1}{2}<A<12\frac{1}{2}$$

By taking smaller and smaller intervals, we are going to bring the lower and upper bounds closer and closer together and so get a better approximation to the actual area.

Example 1 : Estimate the area under the curve $y = x^2$ from $x = 3$ to $x = 6$ by splitting the interval into 3 parts. Note that $y = x^2$ is monotonically increasing in the interval that we are interested in.



The left rectangles give a lower bound on the area:

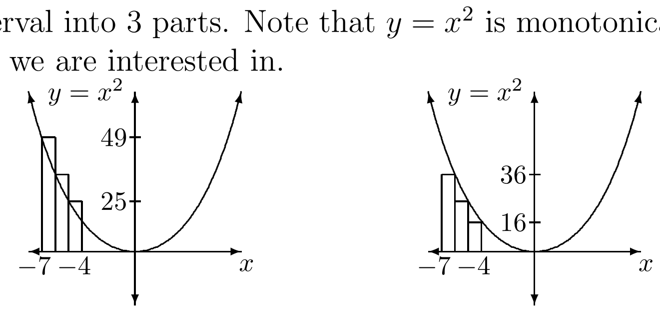
$$A_L = f(3)\times 1 + f(4)\times 1 + f(5)\times 1 = 9+16+25 = 50$$

The right rectangles give an upper bound on the area:

$$A_R = f(4)\times 1 + f(5)\times 1 + f(6)\times 1 = 16+25+36 = 77$$

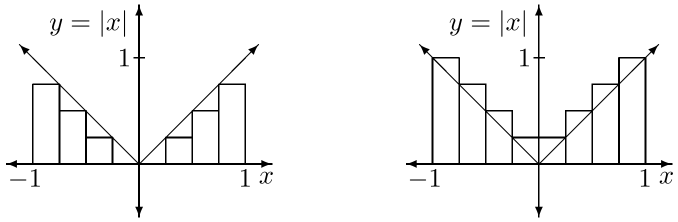
Therefore $50 < A < 77$. This is not a good approximation, but by taking smaller and smaller intervals, the error will be reduced.

Example 2 : Estimate the area under the curve $y = x^2$ from $x = -7$ to $x = -4$ by splitting the interval into 3 parts. Note that $y = x^2$ is monotonically decreasing in the interval that we are interested in.



The left rectangles give $A_L = f(-7)\times 1 + f(-6)\times 1 + f(-5)\times 1 = 49+36+25 = 110$. The right rectangles give $A_R = f(-6)\times 1 + f(-5)\times 1 + f(-4)\times 1 = 36+25+16 = 77$. Notice that A_R is smaller than A_L. This is because now we are looking at a monotonically decreasing function, so A_R sets a lower bound and A_L an upper bound.

Example 3 : Find the area under the curve $y = |x|$ in the interval -1 to 1. Use 8 subintervals.



Since the interval is not monotonically increasing or decreasing, we need to split it up into pieces that are. So first we look at from -1 to 0, which yields the following:

$$A_L(-1,0) = f(-1)\frac{1}{4} + f(-\frac{3}{4})\frac{1}{4} + f(-\frac{1}{2})\frac{1}{4} + f(-\frac{1}{4})\frac{1}{4} = \frac{4+3+2+1}{16} = \frac{5}{8}$$
$$A_R(-1,0) = f(-\frac{3}{4})\frac{1}{4} + f(-\frac{1}{2})\frac{1}{4} + f(-\frac{1}{4})\frac{1}{4} + f(0)\frac{1}{4} = \frac{3+2+1}{16} = \frac{3}{8}$$

The interval from 0 to 1 gives the estimates:

$$A_L(0,1) = f(0)\frac{1}{4} + f(\frac{1}{4})\frac{1}{4} + f(\frac{1}{2})\frac{1}{4} + f(\frac{3}{4})\frac{1}{4} = \frac{3+2+1}{16} = \frac{3}{8}$$
$$A_R(0,1) = f(\frac{1}{4})\frac{1}{4} + f(\frac{1}{2})\frac{1}{4} + f(\frac{3}{4})\frac{1}{4} + f(1)\frac{1}{4} = \frac{4+3+2+1}{16} = \frac{5}{8}$$

To find a lower bound for the area in the given interval, we need to add the two lower bounds together, and similarly for the upper bound. Then the area we require is between $\frac{6}{8}$ and $\frac{10}{8}$. This problem could have been simplified by recognizing that $y = |x|$ is an even function. Then we would only have to double the lower and upper bounds for the area from $x = 0$ to $x = 1$.

Exercises:

1. Using the method described in this section estimate the area under the curve
   (a) $y = x^2$ between $x = 3$ and $x = 6$ using 3 rectangles and finding the upper and lower limits.
   (b) $y = 3x^2 + 1$ between $x = 0$ and $x = 4$ using 8 rectangles and finding the upper and lower limits.
   (c) $y = 4 - x^2$ between $x = -2$ and $x = 0$ using 4 rectangles and finding the upper and lower limits.

Section 2 INTEGRATING POLYNOMIALS

Integration is a technique for finding, amoungst other things, the area under curves. Conceptually, it is like the method of left and right rectangles, but the number of subintervals that the interval of interest is broken up into is infinite, so we get an exact area where the lower and upper bounds are equal. We will not give the details of how one takes the limit of an infinite number of subintervals - we will just state some integration results.

Integration involves anti derivatives, so we will first look at these. The anti derivative of a function f is another function F such that

$$f(x) = F'(x)$$

Thus if $f(x)$ is the derivative of $F(x)$ then $F(x)$ is the anti derivative of $f(x)$. Worksheet 3.8 has an introduction to derivatives. Therefore we can reverse the rules that we had for polynomial differentiation to get anti derivative rules. Recall that if $f(x) = ax^n$, then $f'(x) = anx^{n-1}$. So if $g(x) = bx^m$ then an anti derivative $G(x)$ (such that $G'(x) = g(x)$) is given by

$$G(x) = \frac{b}{m+1}x^{m+1} \quad m \neq -1$$

We add one to the power of x then divide by the new power of x. Note that if

$$\begin{aligned} G(x) &= \frac{b}{m+1}x^{m+1} \\ \text{then} \quad G'(x) &= \frac{b}{m+1}(m+1)x^{m+1-1} \\ &= bx^m \\ &= g(x) \end{aligned}$$

which is what is required. Given $f'(x) = 2x$, then we could have $f(x) = x^2+1$ or $f(x) = x^2+3$ or $f(x) = x^2-4$; notice they differ by the constant term. To compensate for this - the property that the derivative of a constant is zero - we add a constant, usually denoted as c, to the anti derivative. We need more information to find distinct values of c.

Example 1 : Find the anti derivative $F(x)$ of the function $f(x) = 2x + 1$. Note $x^0 = 1$.

$$\begin{aligned} F(x) &= \frac{2x^{1+1}}{2} + \frac{1x^{0+1}}{1} + c \\ &= x^2 + x + c \end{aligned}$$

Example 2 : Find the anti derivative $G(x)$ of the function $g(x) = x^2 + 3x$.

$$\begin{aligned} G(x) &= \frac{x^{2+1}}{3} + \frac{3x^{1+1}}{2} + c \\ &= \frac{x^3}{3} + \frac{3x^2}{2} + c \end{aligned}$$

Example 3 : Find the anti derivative $H(x)$ of the function $h(x) = 5x^4 + 3x^2 + x + x^{-5} + 3$.

$$\begin{aligned} H(x) &= \frac{5x^{4+1}}{5} + \frac{3x^{2+1}}{3} + \frac{x^{1+1}}{2} + \frac{x^{-5+1}}{-4} + \frac{3x^{0+1}}{1} + c \\ &= x^5 + x^3 + \frac{x^2}{2} - \frac{x^{-4}}{4} + 3x + c \end{aligned}$$

Example 4 : Find the anti derivative $F(x)$ of $f(x) = x^{-2}$.

$$F(x) = \frac{x^{-2+1}}{-1} = \frac{x^{-1}}{-1} + c = \frac{-1}{x} + c$$

Example 5 : Find the anti derivative $F(x)$ of $f(x) = 1$.

$$F(x) = \frac{1x^{0+1}}{1} = x + c$$

Example 6 : Find the anti derivative of $f(x) = \frac{3}{x^2} + 4x + 5$. Call the anti derivative $F(x)$.

$$\begin{aligned} f(x) &= 3x^{-2} + 4x + 5 \\ F(x) &= \frac{3x^{-1}}{-1} + \frac{4x^2}{2} + 5x + C \\ &= -\frac{3}{x} + 2x^2 + 5x + C \end{aligned}$$

Exercises:

1. Find the anti derivative of each of the following functions

   (a) $6x^2 + 8x - 3$

   (b) $10x^4 - 3x^2 + 5$

   (c) $3x^4 - 6x^2 - 7$

   (d) $x + 3$

   (e) $x^3 - x^{-3} + 2x + 1$

   (f) $\frac{8}{x^3} - \frac{1}{x^2} + 3x + 4$

   (g) $4x^2 - \frac{7}{x^4} + 2$

   (h) $x^4 - 2x$

   (i) $63x^5 - 1$

   (j) $\frac{4}{x^3} - \frac{6}{x^2}$

Section 3 Integration

The area under the curve $y = f(x)$ between $x = a$ and $x = b$, where $f(x) \geq 0$ for $a \leq x \leq b$, is given by the formula

$$A = \int_a^b f(x)\, dx$$

This is read as the integral of the function $f(x)$ from a to b (where a is taken to be the smaller number). The integral can be evaluated using

$$\int_a^b f(x)\, dx = F(x)\big|_a^b = F(b) - F(a)$$

where $F(x)$ is an anti derivative of $f(x)$.

This is called a definite integral because we integrate between two given values $x = a$ and $x = b$ to obtain a single value. An indefinite integral is written as

$$\int f(x)\, dx = F(x)$$

where again $F(x)$ is an anti derivative of $f(x)$.

<u>Example 1</u> : Calculate $\int 3x^2\, dx$.

$$\begin{aligned} \int 3x^2\, dx &= \frac{3x^{2+1}}{3} + c \\ &= \frac{3x^3}{3} + c \\ &= x^3 + c \end{aligned}$$

We have used the fact that $\int 3x^2\,dx = 3\int x^2\,dx$. In other words, we can 'pull' the 3 through the integral sign because the 3 is independent of the variable that we are integrating with respect to, which is x in this case. In general $\int af(x)\,dx = a\int f(x)\,dx$.

Note: An indefinite integral is the same as calculating the anti derivative.

Example 2 : Calculate $\int_0^1 (x+3)\,dx$.

$$\begin{aligned}\int_0^1 (x+3)\,dx &= \left(\frac{x^{1+1}}{2} + \frac{3x^{0+1}}{1}\right)\Bigg]_0^1 \\ &= \left(\frac{x^2}{2} + 3x\right)\Bigg]_0^1 \\ &= (\frac{1^2}{2} + 3\times 1) - (\frac{0^2}{2} + 3\times 0) \\ &= 3\frac{1}{2}\end{aligned}$$

Example 3 : Calculate the area under the curve $f(x) = x^2$ between $x = 3$ and $x = 6$. The area is given by

$$\begin{aligned}A = \int_3^6 f(x)\,dx &= \int_3^6 x^2\,dx \\ &= \frac{x^3}{3}\Bigg]_3^6 \\ &= F(6) - F(3) \\ &= \frac{6^3}{3} - \frac{3^3}{3} \\ &= 63\end{aligned}$$

Recall that, in example 1 in section 1, we found that the area was between 58 and 77.

Example 4 : Calculate the area under $f(x) = x$ between $x = 5$ and $x = 7$.

$$\begin{aligned}A = \int_5^7 x\,dx &= \frac{x^2}{2}\Bigg]_5^7 \\ &= \frac{49}{2} - \frac{25}{2} \\ &= 12\end{aligned}$$

See the example in section 1 for comparison.

Example 5 : Calculate the area under $f(x) = x^4 + x^2$ between $x = -1$ and $x = 0$.

$$\begin{aligned} A = \int_{-1}^{0} (x^4 + x^2)\, dx &= \left(\frac{x^5}{5} + \frac{x^3}{3}\right)\Bigg]_{-1}^{0} \\ &= \left(\frac{0^5}{5} + \frac{0^3}{3}\right) - \left(\frac{(-1)^5}{5} + \frac{(-1)^3}{3}\right) \\ &= 0 - \left(\frac{-1}{5} - \frac{1}{3}\right) \\ &= \frac{8}{15} \end{aligned}$$

Exercises:

1. Calculate the following integrals

(a) $\int_{-2}^{3} x + 7\, dx$

(b) $\int_{1}^{4} x^2 + 6\, dx$

(c) $\int_{3}^{5} x + 2\, dx$

(d) $\int_{0}^{4} x^2 + x - 1\, dx$

(e) $\int_{-1}^{2} 3x + 4\, dx$

(f) $\int_{0}^{2} 6 - 3x^2\, dx$

(g) $\int_{1}^{3} x^3 - 2x\, dx$

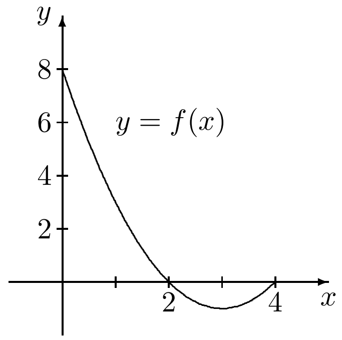
(h) $\int_{0}^{4} x + 2\, dx$

(i) $\int_{-3}^{-1} x^3 + x^2 - 6x\, dx$

(j) $\int_{4}^{6} x + 3\, dx$

Section 4 Integration Continued

As a further investigation of the area under a curve, we will look at the graph of the function $f(x) = x^2 - 6x + 8$.



We will find the area that is shaded. First find the shaded area between $x = 0$ and $x = 2$.

$$\begin{aligned}\int_0^2 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_0^2 \\ &= \left(\frac{8}{3} - 12 + 16\right) - (0 - 0 + 0) \\ &= 6\frac{2}{3}\end{aligned}$$

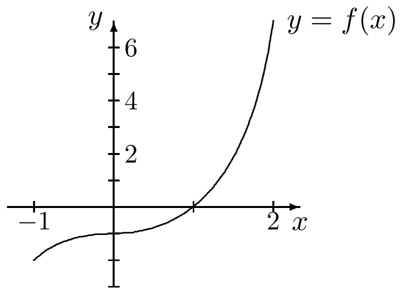
Now see what happens when we use the same method to find the shaded area between $x = 2$ and $x = 4$.

$$\begin{aligned}\int_2^4 x^2 - 6x + 8\,dx &= \left[\frac{x^3}{3} - 3x^2 + 8x\right]_2^4 \\ &= \left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right) \\ &= -1\frac{1}{3}\end{aligned}$$

An area cannot be negative. The negative sign indicates that the region is below the x axis – in this situation, the actual measure of the area is found by taking the absolute value of the integral. That is, the shaded area between $x = 2$ and $x = 4$ is

$$\left|\int_2^4 x^2 - 6x + 8\,dx\right| = \left|-1\frac{1}{3}\right| = 1\frac{1}{3}$$

Example 1 : Find the area bounded by the curve $y = x^3 - 1$, the x axis, and which lies between the lines $x = 0$ and $x = 1$. First draw the graph.



The required area is below the x axis, so

$$\begin{aligned} A &= \left| \int_0^1 x^3 - 1\, dx \right| \\ &= \left| \left[\frac{x^4}{4} - x \right]_0^1 \right| \\ &= \left| \left(\frac{1}{4} - 1 \right) - \left(\frac{0}{4} - 0 \right) \right| \\ &= \left| -\frac{3}{4} \right| \\ &= \frac{3}{4} \end{aligned}$$

<u>Example 2</u> : Find the area bound by the curve $y = x^3 - 1$, the x axis, and the lines $x = 0$ and $x = 3$.

Using the graph from the previous example as a guide, we see that the region from $x = 0$ to $x = 1$ is below the axis, and the region from $x = 1$ to $x = 3$ is above the x axis. So the area we want is

$$\begin{aligned} A &= \left| \int_0^1 x^3 - 1\, dx \right| + \int_1^3 x^3 - 1\, dx \\ &= \left| \left[\frac{x^4}{4} - x \right]_0^1 \right| + \left[\frac{x^4}{4} - x \right]_1^3 \\ &= \left| \left(\frac{1}{4} - 1 \right) - \left(\frac{0}{4} - 0 \right) \right| + \left(\frac{81}{4} - 3 \right) - \left(\frac{1}{4} - 1 \right) \\ &= \left| -\frac{3}{4} \right| + 18 \\ &= 18\frac{3}{4} \end{aligned}$$

Exercises for Worksheet 4.2

1. (a) Use the method of left and right rectangles to find upper and lower bounds for the following functions and integration limits:
   i. $y = \sqrt{x}$ between $x = 0$ and $x = 1$ using 5 subdivisions.
   ii. $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ using 10 subdivisions.

   (b) Find the anti derivative of the following functions:

   i. $f(x) = 1 + x + x^2$ ii. $g(x) = x^{\frac{1}{2}}$ iii. $h(x) = \frac{4}{x^3}$

   (c) Evaluate the following definite integrals:

   i. $\displaystyle\int_0^4 7x\,dx$ ii. $\displaystyle\int_0^1 (1 - y^2)\,dy$ iii. $\displaystyle\int_1^2 3t^2\,dt$

2. (a) By using rectangles of width 1, find the area under $y = [x]$ between $x = 0$ and $x = 5$ where $[x]$ is the 'greatest integer' function e.g. $[3.9] = 3$, $[4.1] = 4$.

   (b) Is the function in (i) monotonically increasing?

   (c) Which is greater, $\int_1^2 x\,dx$ or $\int_1^2 \sqrt{x}\,dx$?

   (d) Calculate the area of the region bounded by the graph of $f(x) = (x-2)^2$, the x-axis, and between $x = 2$ and $x = 3$.

   (e) Calculate the area bounded by the curve $y = x^2(3 - x)$ and the x-axis.

   (f) If $\displaystyle\int_{-1}^{a} x\,dx = 0$, evaluate a.

   (g) If $c\displaystyle\int_{-2}^{2}(x - 5)\,dx = 1$, evaluate c.

3. (a) Calculate the area bound by the curves $f(x) = \frac{x^2}{4} - 2$ and $g(x) = x + 1$.
   (Hint: Find the points of intersection of the two curves, and calculate both areas.)

   (b) Show, by integration, that the area of a unit square is:
   (a) Bisected by the line $y = x$.
   (b) Trisected by the curves $y = x^2$ and $y = \sqrt{x}$.

   (c) The marginal revenue, MR, that a manufacturer receives for his goods is given by $MR = \frac{dR}{dq} = 100 - 0.03q$. Find the total revenue function $R(q)$.

   (d) The density curve of a 10-metre beam is given by $\rho(x) = 3x + 2x^2 - x^{\frac{3}{2}}$ where x is the distance measured from one edge of the beam. The mass of the beam is calculated to be the area under the curve $\rho(x)$ between 0 and x. Find the mass of the beam.

Worksheet 4.3 Integrating Special Functions

Section 1 Exponential and Logarithmic Functions

Recall from worksheet 3.10 that the derivative of e^x is e^x. It then follows that the anti derivative of e^x is e^x:

$$\int e^x\,dx = e^x + c$$

In worksheet 3.10 we also discussed the derivative of $e^{f(x)}$ which is $f'(x)e^{f(x)}$. It then follows that

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

where $f(x)$ can be any function. There are other ways of doing such integrations, one of which is by substitution.

Example 1 : Evaluate the indefinite integral $\int 3e^{3x+2}\,dx$.

We recognize that $3 = \frac{d(3x+2)}{dx}$ so that the expression we are integrating has the form $f'(x)e^{f(x)}$. Then

$$\int 3e^{3x+2}\,dx = e^{3x+2} + c$$

Alternatively, we could do it by substitution: let $u = 3x + 2$. Then $du = 3dx$, and

$$\int 3e^{3x+2}\,dx = \int e^u\,du = e^u = e^{3x+2}$$

Note that the integral of the function e^{ax+b} (where a and b are constants) is given by

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

Example 2 : Find the area under the curve $y = e^{5x}$ between 0 and 2.

$$\begin{aligned}
A &= \int_0^2 e^{5x}\,dx \\
&= \left.\frac{1}{5}e^{5x}\right]_0^2 \\
&= \frac{1}{5}e^{10} - \frac{1}{5}e^0 \\
&= \frac{1}{5}(e^{10} - 1)
\end{aligned}$$

We used the property that for any real number x, $x^0 = 1$.

Recall that the derivative of $\log_e x$ is $\frac{1}{x}$. Then the anti derivative of $\frac{1}{x}$ is $\log_e x$. Notice that $\frac{1}{x} = x^{-1}$, and that if we had used the rules we have developed to find the anti derivatives of things like x^m, we would have the anti derivative of x^{-1} being $\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0}$ which is not defined as we can not divide by zero. So we have the special rule for the anti derivative of $1/x$:

$$\int \frac{1}{x}\,dx = \log_e x + c$$

Recall that the derivative of $\log_e f(x)$ is $\frac{f'(x)}{f(x)}$. Then we have

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + c$$

<u>Example 3</u> : Evaluate the indefinite integral $\int \frac{5}{5x+2}\,dx$. This has the form $\int \frac{f'(x)}{f(x)}\,dx$ so we get

$$\int \frac{5}{5x+2}\,dx = \log_e(5x+2) + c$$

Note that when you need to integrate a function like $1/(ax+b)$ (where a and b are constants), then

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{a}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c$$

<u>Example 4</u> : Find the area under the curve $f(x) = 1/(2x+3)$ between 3 and 11.

$$\begin{aligned} A &= \int_3^{11} \frac{1}{2x+3}\,dx \\ &= \frac{1}{2}\log_e(2x+3)\Big]_3^{11} \\ &= \frac{1}{2}\log_e(2\times 11+3) - \frac{1}{2}\log_e(2\times 3+3) \\ &= \frac{1}{2}\log_e 25 - \frac{1}{2}\log_e 9 \\ &= \log_e(25)^{\frac{1}{2}} - \log_e(9)^{\frac{1}{2}} \\ &= \log_e \frac{5}{3} \end{aligned}$$

Section 2 Integrating Trig Functions

To integrate trig functions we need to recall the derivatives of trig functions. We can then work out the anti derivatives of $\cos x$, $\sin x$, and $\sec^2 x$. For more complicated integrals we need special techniques that you will learn in first-year maths. The derivatives of the trig functions are:

$$\begin{array}{rclrcl} g(x) & = & \sin(ax+b) & g'(x) & = & a\cos(ax+b) \\ f(x) & = & \cos(ax+b) & f'(x) & = & -a\sin(ax+b) \\ h(x) & = & \tan(ax+b) & h'(x) & = & a\sec^2(ax+b) \end{array}$$

Example 1 : Evaluate the indefinite integral $\int \sin 3x\, dx$.

$$\int \sin 3x\, dx = \frac{-1}{3}\cos 3x + c$$

Note : A good way of checking your answers to indefinite integrals is to differentiate them. You should recover the function that you started with.

Example 2 : Find the area under the curve $y = \cos x$ between 0 and $\frac{\pi}{2}$.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \cos x\, dx \\ &= \sin x\Big]_0^{\frac{\pi}{2}} \\ &= \sin\frac{\pi}{2} - \sin 0 \\ &= 1 \text{ square units} \end{aligned}$$

Example 3 : Find $\int f(x)\, dx$ if $f(x) = -3\sin(3x+2)$.

$$\int -3\sin(3x+2)\, dx = \cos(3x+2) + c$$

Example 4 : What is the area under the curve $y = \sec^2\frac{x}{2}$ between $\frac{\pi}{2}$ and 0?

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \sec^2\frac{x}{2}\, dx \\ &= \frac{1}{1/2}\tan\frac{x}{2}\Big]_0^{\frac{\pi}{2}} \end{aligned}$$

$$
\begin{aligned}
&= 2\tan\frac{x}{2}\Big]_0^{\frac{\pi}{2}} \\
&= 2\tan\frac{\pi}{4} - 2\tan 0 \\
&= 2 - 0 \\
&= 2 \quad \text{square units}
\end{aligned}
$$

<u>Example 5</u> : Evaluate the indefinite integral $\int 5\sec^2 5x\,dx$.

$$\int 5\sec^2 5x\,dx = \tan 5x + c$$

Exercises for Worksheet 4.3

1. (a) Find the anti derivative of

   i. e^{-4x}

   ii. $\sqrt{e^x}$

   iii. $\dfrac{7-6x}{8+7x-3x^2}$

   iv. $\cos 2x$

   v. $\sec^2(5x-2)$

   vi. $\dfrac{1-x}{x^2}$

   (b) Evaluate

   i. $\displaystyle\int_0^{\frac{1}{2}} e^{2x}\,dx$

   ii. $\displaystyle\int_{-1}^{1} \frac{2x+1}{x^2+x+1}\,dx$

   iii. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

   iv. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \cos x\,dx$

2. (a) Calculate the area under the curve $y = \frac{2}{x+3}$ from $x = 2$ to $x = 3$.

   (b) Calculate the area under the curve $y = e^{3x}$ from $x = 0$ to $x = 3$.

   (c) The area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = b$ is 1 unit. What is b?

   (d) Find the points of intersection of the curve $y = \sin x$ with the line $y = \frac{1}{2}$ and hence find the area between the two curves (from one intersection to the next). There are two possible areas you can end up with; choose the one above $y = \frac{1}{2}$.

   (e) Show, by simple division, that $\dfrac{x+6}{x+2} = 1 + \dfrac{4}{x+2}$. Hence evaluate $\displaystyle\int \frac{x+6}{x+2}\,dx$.

Worksheet 4.4 Applications of Integration

Section 1 Movement

Recall that the derivative of a function tells us about its slope. What does the slope represent? It is the change in one variable with respect to the other variable. Say a line has a constant slope of 4; then for every 1 unit change in x, there will be a 4 unit change in y. Say we had a function that represented the movement of a car, so that the distance was plotted as a function of time. The change in distance over a small amount of time would represent the speed of the car. Thus in this case the slope of the function represents the speed of the car, and is given by $\frac{dx}{dt}$. The rate of change in speed of the car is called acceleration and this is given by $\frac{d}{dt}(\frac{dx}{dt}) = \frac{d^2x}{dt^2}$.

So if we have a function $x = f(t)$ that represents distance as a function of time, then $\frac{dx}{dt}$ is the speed and $\frac{d^2x}{dt^2}$ is the acceleration. Conversely, if we have a function that represents the velocity of a vehicle and we integrate it we get the distance travelled as a function of time.

The methods of integration and differentiation can be used to solve problems involving movement.

Example 1 : If the velocity of a particle is given by $v = 3t^2$, what is the distance travelled as a function of time? Since $v = \frac{dx}{dt}$, where x is the distance, the anti derivative of v will tell us the distance.

$$x = \frac{3t^3}{3} + c = t^3 + c$$

Example 2 : If the velocity of a particle is given as $v = 3t^2$ metres per second, what is the distance travelled between $t = 0$ and $t = 2$? In example 1, we worked out that the distance travelled was $x = t^3 + c$. Therefore

$$\text{Distance } = (t^3 + c)\Big]_0^2 = (2^3 + c) - (0^3 + c) = 8 \text{ metres}$$

The particle covers a distance of 8 metres between $t = 0$ and $t = 2$.

Section 2 Initial-Value Problems

Recall that, when working out anti derivative problems, there is a constant of integration that is undetermined (and which we have usually denoted by c). Initial-value problems ask us to find anti derivatives which take on specific values at certain points so that we can determine the value of the constant.

Example 1 : If $\frac{dx}{dt} = 5$ and $x = 9$ when $t = 0$, what is x as a function of time? As $\frac{dx}{dt} = 5$, then $x = 5t + c$. Using the information that when $t = 0$, $x = 9$ we can now write an equation to solve for c:

$$9 = 5 \times 0 + c$$

which has the solution $c = 9$. The complete solution for x is $x = 5t + 9$.

Example 2 : The acceleration of a car is given by $a = 6t$ and the velocity is 2 when $t = 0$, and the distance from home is 1 when $t = 0$. What is the distance from home as a function of time? Acceleration is the derivative of velocity, so velocity is the anti derivative of acceleration. Then

$$v = \int 6t \, dt = 3t^2 + c_1$$

When $t = 0$, $v = 2$, so that we can write down an equation for c_1 and solve it: $2 = 0 + c_1$. Therefore, $c_1 = 2$. The velocity is then $v = 3t^2 + 2$. Distance is the anti derivative of velocity, which gives

$$x = \int (3t^2 + 2) \, dt = t^3 + 2t + c_2$$

Using $x = 1$ when $t = 0$, we can write down an equation for c_2: $1 = 0 + 0 + c_2$. Therefore $c_2 = 1$, and so the distance as a function of time is

$$x = t^3 + 2t + 1$$

Example 3 : If $\frac{d^2x}{dt^2} = 3$, and when $t = 0$ we have $\frac{dx}{dt} = 0$ and $x = 0$, what is x as a function of t?

$$\begin{aligned} \frac{d^2x}{dt^2} &= 3 \\ \frac{dx}{dt} &= 3t + c \end{aligned}$$

Using the information we are given for $\frac{dx}{dt}$ at $t = 0$, we find $3 \times 0 + c = 0$ so that $c = 0$. Then $\frac{dx}{dt} = 3t$ for all t. The anti derivative of this will give us x:

$$x = \frac{3t^2}{2} + c$$

Using the information for x at $t = 0$, we find $c = 0$, so that $x = \frac{3t^2}{2}$ for all t.

Section 3 Application to Growth

Exponential functions are used to represent the growth and decay of populations and radioactive elements, among other things. We can use a general form of an equation for exponential growth or decay and we find a specific equation which uses initial values as in the application of integration to motion.

Exponential growth and decay is represented by the equation $P(t) = P(0)e^{kt}$ where $P(t)$ is the population at time t, $P(0)$ is the population at $t = 0$, and k is some constant which depends on the population being looked at. A similar formula applies to the decay of radioactive material. $P(0)$ would then represent the amount of radioactive material at $t = 0$.

Example 1 : What is $P(10)$ if $P(0) = 100$ and $k = 1$ given $P(t) = P(0)e^{kt}$.

$$\begin{aligned} P(10) &= P(0)e^{kt} \\ &= 100e^{10} \end{aligned}$$

Example 2 : If $P(10) = 1000$ and $P(0) = 100$, what is k in the expression $P(t) = P(0)e^{kt}$? We put $t = 10$ into the equation for $P(t)$ and equate this to what we are given at $P(10)$.

$$P(10) = 1000 = 100e^{k10}$$

This equation can be solved for k:

$$\begin{aligned} 1000 &= 100e^{10k} \\ 10 &= e^{10k} \\ \log 10 &= 10k \\ k &= \frac{1}{10}\log 10 \end{aligned}$$

Example 3 : If the growth constant for a population of bees is $\frac{1}{10}$ and the initial population of a hive is 75, what is the population at time t?

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ &= 75e^{\frac{1}{10}t} \end{aligned}$$

Notice that if $k > 0$ the population is growing but if $k < 0$ the population is getting smaller.

Example 4 : For what values of k does the population $P(t) = P(0)e^{kt}$ remain constant? We need $P(t) = P(0)$ for all t. Then

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ 1 &= e^{kt} \end{aligned}$$

This is true when $k = 0$.

Exercises for Worksheet 4.4

1. The derivatives of a function and one point on its graph are given. Find the function.

   (a) $\dfrac{dy}{dx} = x^3 + x^2 - 3$; $(1, 5)$

   (b) $\dfrac{dy}{dx} = 2x(x + 1)$; $(2, 0)$

   (c) $y' = \cos x$; $(\frac{\pi}{6}, 4)$

   (d) $y' = \dfrac{x}{\sqrt{10 - x^2}}$; $(1, 5)$

2. (a) Find $f(x)$ if its gradient function is $2x - 2$ and $f(1) = 4$.

   (b) The velocity $v(t)$ of a particle moving in a straight line is given by $v(t) = 12t^2 - 6t + 1$, $t \geq 0$. Find its position coordinate $s(t) = \int v(t)\,dt$ given that $s(1) = 4$.

   (c) If $\frac{dx}{dt} = kx$ and $x = 10$ when $t = 0$,

      i. Show that $x = 10e^{kt}$.

      ii. Find k if $x = 20$ when $t = 10$.

   (d) A radioactive substance decays according to the rule $\frac{dM}{dt} = -0.2M$. If $M = 5$ when $t = 0$,

      i. Show that $M = 5e^{-0.2t}$.

      ii. Find M when $t = 5$.

   (e) A ship travelling at 10 metres per second is subjected to water resistance proportional to the speed. The engines are cut and the ship slows down according to the rule $\frac{dv}{dt} = -kv$.

      i. Show that the velocity after t seconds is given by $v = 10e^{-kt}$ metres per second.

      ii. If, after 20 seconds, $v = 5$m/s, find k.

3. (a) A particle moves with constant acceleration of 5.8 metres/second squared. It starts with an initial velocity of 0.2m/s, and an initial position of 25m. Find the equation of motion of the particle given $\int$ (acceleration) dt = velocity, and $\int$ (velocity) dt = position.

   (b) If the instantaneous rate of change of a population is $50t^2 - 100t^{\frac{3}{2}}$ (measured in individuals per year) and the initial population is 25000 then

      (a) What is the population after t years?

      (b) What is the population after 25 years?

   (c) A particle moves along a straight line with an acceleration of $a = 4\sin\frac{\pi t}{2}$ m/s^2. If the displacement at $t = 0$ is 0, and the initial velocity is $-\frac{8}{\pi}$m/s, find

      i. The acceleration after 2 seconds.

      ii. The velocity after 2 seconds.

      iii. The displacement after 2 seconds.

Worksheet 4.5 Binomial Coefficients

Section 1 Introduction

We wish to be able to expand an expression of the form $(ax + b)^n$. We can do this easily for $n = 2$, but what about a large n? It would be tedious to manually multiply $(ax + b)$ by itself 10 times, say. There are two methods of expanding an expression of this type without doing all the multiplications involved. The first method we will look at is called Pascal's triangle. For a given n, this method uses the expansion of $n - 1$. The first 5 rows of Pascal's triangle are shown:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Pascal's triangle is particularly useful when dealing with small n. The triangle is easy to remember how to reproduce as each entry is the sum of the two right and left entries on the line above, and the sides are always one. Thus for the second entry of line five we get

1 3
\ /
4

For the expansion of $(x+1)^2$, we use the third line of the triangle. For the expansion of $(x+1)^n$, we need the $(n + 1)$th line of Pascal's triangle, and we have the following rule: the 1st term in the line is the coefficient of x^n, the second term in the line is the coefficient of x^{n-1}, and so on until the $(n + 1)$th term in the line (the last term) is the coefficient of $1^n = 1$. If you wish to use Pascal's triangle on an expansion of the form $(ax + b)^n$, then some care is needed. The $(n+1)$th row is the row we need, and the 1st term in the row is the coefficient of $(ax)^n b^0$. The second term in the row is the coefficient of $(ax)^{n-1}b^1$. The last term in the row - the $(n + 1)$th term - is the coefficient of $(ax)^0 b^n$. Care should be taken when minus signs are involved.

Example 1 : Expand $(1 + x)^4$. Using the coefficients in the fifth row,

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

Example 2 : What is the coefficient of x^2 in $(x + 2)^5$? The 6th line from Pascal's triangle is 1 5 10 10 5 1. The x^2 term is the 4th term in the expansion, so we pick up 10 from Pascal's triangle. But the expression we are raising to the power 5 is of the form $(ax + b)$ with $a = 1$ and $b = 2$. So the fourth term also has a factor a^2b^3 which in this case is $2^3 = 8$. So the coefficient of x^2 is $10 \times 8 = 80$.

Example 3 : Expand the expression $(ax + b)^3$. The 4th row of Pascal's triangle is 1 3 3 1. So

$$\begin{aligned}(ax+b)^3 &= (ax)^3 + 3(ax)^2b + 3(ax)^1b^2 + b^3 \\ &= a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3\end{aligned}$$

Notice that the powers of a and b in each term always add to give the power of the expansion. This is always the case.

Example 4 : Find the constant term (the term that is independent of x) in the expansion of $(x - 2)^5$. The constant term is the last term, and is $(-2)^5$. Notice that the minus sign is important.

Example 5 : Find the 4th term in the expansion of $(2x - 3)^5$. The 4th term in the 6th line of Pascal's triangle is 10. So the 4th term is

$$10(2x)^2(-3)^3 = -1080x^2$$

The 4th term is $-1080x^2$.

The second method to work out the expansion of an expression like $(ax + b)^n$ uses binomial coefficients. This method is more useful than Pascal's triangle when n is large. To work out binomial coefficients, we need to know what $n!$ - which is read as n factorial - means. It is defined by

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

In addition, we define $0! = 1$.

Example 6 : What is 4!?

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Example 7 : What is 5!?

$$5! = 5 \times 4! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Example 8 : Evaluate $\frac{7!}{5!}$.

$$\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$$

Example 9 : Evaluate $\frac{n!}{(n-3)!3!}$.

$$\begin{aligned} \frac{n!}{(n-3)!3!} &= \frac{n \times (n-1) \times \ldots \times 2 \times 1}{(n-3) \times (n-4) \times \ldots \times 2 \times 1 \times 3 \times 2 \times 1} \\ &= \frac{n \times (n-1) \times (n-2)}{6} \end{aligned}$$

The binomial coefficients can be used to find the expansion of $(ax+b)^n$ - the coefficient of x^k is given by

$$\frac{n!}{(n-k)!k!} a^k b^{n-k}$$

where $\frac{n!}{(n-k)!k!}$ is called the binomial coefficient.

Example 10 : What is the coefficient of x^{10} in the expansion of $(x+1)^{12}$?

$$\frac{12!}{(12-10)!10!} 1^{10} 1^2 = \frac{12!}{10!2!} = 66$$

The coefficient of x^{10} in $(x+1)^{12}$ is 66.

Example 11 : What is the coefficient of x in the expansion of $(2x-1)^8$?

$$\frac{8!}{(8-1)!1!} 2^1 (-1)^7 = -\frac{8!}{7!} = -16$$

Example 12 : What will be the coefficient of x^5 in $\frac{1}{x^2}(2x-3)^9$? This will be the same as the coefficient of x^7 in $(2x-3)^9$, which is given by

$$\frac{9!}{(9-7)!7!} \times 2^7 \times (-3)^{9-7} = \frac{9!}{2!7!} 2^7 (-3)^2 = 81 \times 8 \times 2^6 = 3^4 \times 2^9$$

The answer can be left like this or, if you prefer, you can calculate further.

Exercises for Worksheet 4.5

1. (a) Evaluate 9!
   (b) Evaluate $\dfrac{5!}{2!3!}$
   (c) Show that ${}^5C_2 = {}^5C_3$, where ${}^nC_k = \frac{n!}{k!(n-k)!}$.
   (d) Prove that ${}^nC_k = {}^nC_{n-k}$
   (e) Prove that $\dfrac{(2n)!}{2^n n!} = \dfrac{1}{2}^n \, {}^{2n}C_n n!$
   (f) Write down expansions of the following:
      i. $(2x+3y)^4$
      ii. $(a+\frac{1}{a})^6$
      iii. $(\frac{a}{b}-\frac{b}{a})^7$
      iv. $(x^2+a)^5$

2. (a) Write down and simplify the 4th term of
      i. $(\frac{m}{2}+3n)^8$
      ii. $(x-\frac{1}{x})^n$
   (b) Find the coefficients of
      i. x^2 in $(x+\frac{1}{x})^8$
      ii. a^5b^4 in $(3a-\frac{b}{3})^9$
   (c) Find the constant terms in
      i. $(2x-\frac{1}{x^2})^9$
      ii. $(2x+\frac{1}{x})^{2n}$
   (d) Find the largest positive numerical term in the expansion of $(1-2x)^9$ if $x=3$

3. (a) Express $(1.08)^3$ as a binomial of the form $(a+b)^n$, and evaluate it.
   (b) The number of ways of choosing k objects from n objects is given by ${}^nC_k$.
      i. How many different sets of three colours can be selected from the colours red, orange, yellow, green, blue, and violet?
      ii. In how many ways can a team of five basketball players be selected from 8 boys?
      iii. A secretary has nine letters and only five stamps. How many ways can he select the letters for posting?
      iv. In a plane, there are 5 points, no three of which are collinear. How many different triangles can be drawn by joining sets of three points?

Worksheet 4.6 Sigma Notation

Section 1 Introduction to Sigma Notation

Sigma notation is used as a convenient shorthand notation for the summation of terms.

Example 1 : We write

$$\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5.$$

Here the symbol $\sum$ (sigma) indicates a sum. The numbers at the top and bottom of sigma are called boundaries and tell us what numbers we substitute in to the expression for the terms in our sum. What comes after sigma is an algebraic expression representing terms in the sum. In the example above, n is a variable and represents the terms in our sum.

Example 2 :

$$\sum_{n=1}^{5} n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Example 3 :

$$\sum_{n=3}^{5} n^3 = 3^3 + 4^3 + 5^3.$$

Example 4 :

$$\sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

Note that we have $\displaystyle\sum_{n=1}^{5} n = \sum_{i=1}^{5} i$. The n and the i just play the role of dummy variables.

We can also work the other way. Sometimes our sum has a pattern which enables us to write the sum using sigma notation.

Example 5 : Write the expression $3 + 6 + 9 + 12 + \cdots + 60$ in sigma notation.

- notice that we are adding multiples of 3;
- so we can write this sum as $\sum_{n=1}^{20} 3n$.

<u>Example 6</u> : Write the expression $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots + \frac{1}{3n+1}$ in sigma notation.

- notice that we are adding fractions with a numerator of 1 and denominators starting with 1 in the first term and then increasing by 3 in each subsequent term;
- i.e. the denominator can be represented by $3k + 1$ for $k = 0, 1, \ldots, n$;
- so we can write this sum as $\sum_{k=0}^{n} \frac{1}{3k+1}$.

We can also use sigma notation when we have variables in our terms.

<u>Example 7</u> : Write the expression $3x+6x^2+9x^3+12+\cdots+60x^{20}$ in sigma notation.

- note from Example 5 the numbers are multiples of 3 and can be represented by $3n$ where $n = 1, 2, \ldots, 20$;
- we also have powers of x which increase by 1 in each subsequent term;
- so we can write this sum as $\sum_{n=1}^{20} 3nx^n$.

The numbers in front of the variables are called coefficients. In Example 7 the coefficient of x is 3 and the coefficient of x^2 is 6.

<u>Example 8</u> : Write the expression $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!}$ in sigma notation.

- here the powers of x are even numbers which can be represented by $2k$ for $k = 0, 1, \ldots, n$;
- the denominators are also even numbers but with factorials;
- so we can write this sum as $\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$.

Exercises:

1. Write out each of the following sums.

(a) $\displaystyle\sum_{n=1}^{6} n^4$

(b) $\displaystyle\sum_{k=3}^{7} \frac{k+1}{k}$

(c) $\displaystyle\sum_{i=2}^{n} (2i-1)$

(d) $\displaystyle\sum_{k=0}^{n} 2^{k+1}x^k$

(e) $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^k}{2k+1}$

2. Express each of these sums using sigma notation.

(a) $1+4+9+16+25+36$

(b) $3-5+7-9+11-13+15$

(c) $\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}$

(d) $\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots+\frac{n+1}{n+2}$

(e) $2-2^2+2^3-2^4+\cdots+2^{2n+1}$

(f) $2x^3+4x^5+6x^7+\cdots+30x^{31}$

(g) $x-x^2+\frac{x^3}{2!}-\frac{x^4}{3!}+\frac{x^5}{4!}-\frac{x^6}{5!}$

(h) $3x+7x^2+11x^3+15x^4+19x^5+23x^6$

(i) $8x^4+10x^5+12x^6+\cdots+(2n+2)x^{n+1}$

(j) $12x^4+20x^5+30x^6+\cdots+n(n-1)x^n$

(k) $\frac{x^3}{2}-\frac{x^5}{3}+\frac{x^7}{4}-\frac{x^9}{5}+\cdots-\frac{x^{199}}{100}$

Section 2 Finding Coefficients

Sigma notation is a useful way to express the sum of a large number of terms. When we want to find particular terms or coefficients, we don't always have to expand the whole expression to find it.

Example 1 : Find the coefficient of x^4 in $\displaystyle\sum_{k=0}^{8}(4k+3)x^k$.

- the terms in this sum look like $(4k+3)x^k$;
- the terms with x^4 occurs when $k=4$ i.e. $(4(4)+3)\,x^4 = 19x^4$;
- the coefficient of x^4 is 19.

Example 2 : Find the coefficient of x^7 in $\displaystyle\sum_{k=0}^{8}(4k+3)x^{k+2}$.

- a typical term is of the form $(4k+3)x^{k+2}$;

- the term with x^7 occurs when $k+2=7$ i.e. $k=5$;
- we have $(4(5)+3)\,x^{5+2} = 23x^7$;
- the coefficient of x^7 is 23.

Example 3 : Find the coefficient of x^2 in $(3+x)\sum_{k=0}^{8}(4k+3)x^k$.

- we can think of this as $3\sum_{k=0}^{8}(4k+3)x^k + x\sum_{k=0}^{8}(4k+3)x^k$;
- the term with x^2 can be obtained by taking $k=2$ from the first part of this expression to get $3\,(4(2)+3)\,x^2 = 33x^2$ and then taking $k=1$ from the second part of this expression to get $x\,(4(1)+3)\,x^1 = 7x^2$;
- combining these we get $33x^2 + 7x^2 = 40x^2$;
- so the coefficient of x^2 is 40.

Exercises:

1. Find the coefficients of x^2 and x^6 in the following.

(a) $\displaystyle\sum_{r=0}^{10}\frac{r+1}{r!}x^r$

(b) $\displaystyle\sum_{k=3}^{15}k(k+1)x^{k-2}$

(c) $\displaystyle\sum_{n=0}^{20}\frac{(-1)^n x^{4n+2}}{n+3}$

(d) $\displaystyle(3+2x)\sum_{k=0}^{8}(k+1)x^k$

(e) $\displaystyle(1-x)\sum_{k=0}^{7}\frac{x^{k+1}}{k!}$

(f) $\displaystyle(x+x^2)\sum_{k=0}^{15}(2k+1)x^k$

Exercises for Worksheet 4.6

1. Write out each of the following sums.

   (a) $\displaystyle\sum_{r=0}^{7} r^2(-x)^r$ (b) $\displaystyle\sum_{k=3}^{8} \frac{k-1}{k+1}x^{2k}$ (c) $\displaystyle\sum_{k=1}^{n+1} k(k-1)x^{3k}$

2. Write each of the following series in sigma notation.

   (a) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$

   (b) $7 - 10 + 13 - 16 + \cdots + 31$

   (c) $4^2 + 5^2 + 6^2 + 7^2 + \cdots + (n+2)^2$

   (d) $\frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \cdots + \frac{1}{(n+5)!}$

   (e) $3x^2 + 6x^4 + 9x^6 + 12x^8 + \cdots + 36x^{24}$

   (f) $x^7 + \frac{x^9}{1!} + \frac{x^{11}}{2!} + \frac{x^{13}}{3!} + \cdots + \frac{x^{31}}{12!}$

   (g) $x - 5x^2 + 9x^3 - 13x^4 + \cdots - 41x^{10}$

   (h) $\frac{5}{2}x^3 + 3x^4 + \frac{7}{2}x^5 + 4x^6 + \cdots + \frac{n}{2}x^{n-2}$

   (i) $6x^{12} + 7x^{14} + 8x^{16} + 9x^{18} + \cdots + (n+1)x^{2n+2}$

3. Find the coefficient of x, x^3, and x^7 in the following expressions.

   (a) $\displaystyle\sum_{k=3}^{n} \frac{(-1)^k x^{k-2}}{(4k-1)!}$

   (b) $\displaystyle\sum_{k=1}^{n} \frac{(3x)^{k-3}}{(k+1)^2}$

   (c) $\displaystyle(2+x)\sum_{k=0}^{n} \frac{k-1}{k+1}x^k$

   (d) $\displaystyle(x-x^2)\sum_{k=0}^{n} \frac{(-1)^{k+1}}{3k!}x^k$

   (e) $\displaystyle(5+x^2)\sum_{k=1}^{n} \frac{k}{(2k+1)!}x^{k-2}$

4. Simplify the following expressions.

   (a) $\displaystyle\sum_{k=0}^{n} k^2 - (k+1)^2$ (b) $\displaystyle\sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k}\right)$

Worksheet 4.7 Polynomials

Section 1 Introduction to Polynomials

A polynomial is an expression of the form

$$p(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n \quad (n \in \mathbb{N})$$

where $p_0, p_1, \ldots, p_n$ are constants and x is a variable.

Example 1 : $f(x) = 3 + 5x + 7x^2 - x^4 \quad (p_0 = 3,\ p_1 = 5,\ p_2 = 7,\ p_3 = 0,\ p_4 = -1)$

Example 2 : $g(x) = 2x^3 + 3x \quad (p_0 = 0,\ p_1 = 3,\ p_2 = 0,\ p_3 = 2)$

- The constants $p_0, \ldots, p_n$ are called coefficients. In Example 1 the coefficient of x is 5 and the coefficient of x^4 is -1. The term which is independent of x is called the constant term. In Example 1 the constant term of $f(x)$ is 3; in Example 2 the constant term of $g(x)$ is 0.
- A polynomial $p_0 + p_1x + \cdots + p_nx^n$ is said to have degree n, denoted $\deg n$, if $p_n \neq 0$ and x^n is the highest power of x which appears. In Example 1 the degree of $f(x)$ is 4; in Example 2 the degree of $g(x)$ is 3.
- A zero polynomial is a polynomial whose coefficients are all 0 i.e. $p_0 = p_1 = \cdots = p_n = 0$.
- Two polynomials are equal if all the coefficients of the corresponding powers of x are equal.

Exercises:

1. Find (i) the constant term, (ii) the coefficient of x^4 and (iii) the degree of the following polynomials.
   (a) $x^4 + x^3 + x^2 + x + 1$
   (b) $9 - 3x^2 + 7x^3$
   (c) $x - 2$
   (d) $10x^5 - 3x^4 + 5x + 6$
   (e) $3x^6 + 7x^4 + 2x$
   (f) $7 + 5x + x^2 - 6x^4$
2. Suppose $f(x) = 2ax^3 - 3x^2 - b^2x - 7$ and $g(x) = cx^4 + 10x^3 - (d+1)x^2 - 4x + e$. Find values for constants a, b, c, d and e given that $f(x) = g(x)$.

Section 2 Operation on Polynomials

Suppose $h(x) = 3x^2 + 4x + 5$
$k(x) = 7x^2 - 3x$

- Addition/Subtraction: To add/subtract polynomials we combine like terms.

  Example 1 :We have
  $$h(x) + k(x) = 10x^2 + x + 5$$

- Multiplication: To multiply polynomials we expand and then simplify their product.

  Example 2 :We have
  $$\begin{aligned} h(x) \cdot k(x) &= (3x^2 + 4x + 5)(7x^2 - 3x) \\ &= 21x^4 - 9x^3 + 28x^3 - 12x^2 + 35x^2 - 15x \\ &= 21x^4 + 19x^3 + 23x^2 - 15x \end{aligned}$$

- Substitution: We can substitute in different values of x to find the value of our polynomial at this point.

  Example 3 :We have
  $$\begin{aligned} h(1) &= 3(1)^2 + 4(1) + 5 = 12 \\ k(-2) &= 7(-2)^2 - 3(-2) = 34 \end{aligned}$$

- Division: To divide one polynomial by another we use the method of long division.

  Example 4 :Suppose we wanted to divide $3x^3 - 2x^2 + 4x + 7$ by $x^2 + 2x$.
  $$\begin{array}{r|l} & 3x - 8 \\ \hline x^2 + 2x & 3x^3 - 2x^2 + 4x + 7 \\ & \underline{3x^3 + 6x^2} \\ & \quad -8x^2 + 4x \\ & \quad \underline{-8x^2 - 16x} \\ & \qquad\qquad 20x + 7 \end{array}$$

  So $3x^3 - 2x^2 + 4x + 7$ divided by $x^2 + 2x$ gives us a quotient of $3x - 8$ with a remainder of $20x + 7$. We have
  $$3x^3 - 2x^2 + 4x + 7 = (x^2 + 2x)(3x - 8) + (20x + 7).$$

More formally, suppose $p(x)$ and $f(x)$ are polynomials where $\deg p(x) \geq \deg f(x)$. Then dividing $p(x)$ by $f(x)$ gives us the identity

$$p(x) = f(x)q(x) + r(x),$$

where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg r(x) < \deg f(x)$.

Example 5 :Dividing $p(x) = x^3 - 7x^2 + 4$ by $f(x) = x - 1$ we obtain the following result:

$$\begin{array}{r|l} & x^2 - 6x - 6 \\ \hline x - 1 & x^3 - 7x^2 + 0x + 4 \\ & \underline{x^3 - x^2} \\ & \quad -6x^2 + 0x \\ & \quad \underline{-6x^2 + 6x} \\ & \qquad\qquad -6x + 4 \\ & \qquad\qquad \underline{-6x + 6} \\ & \qquad\qquad\qquad -2 \end{array}$$

Here the quotient is $q(x) = x^2 - 6x - 6$ and the remainder is $r = -2$. Note: As we can see, division doesn't always produce a polynomial annswer- sometimes there's just a constant remainder.

Exercises:

1. Perform the following operations and find the degree of the result.
   (a) $(2x - 4x^2 + 7) + (3x^2 - 12x - 7)$
   (b) $(x^2 + 3x)(4x^3 - 3x - 1)$
   (c) $(x^2 + 2x + 1)^2$
   (d) $(5x^4 - 7x^3 + 2x + 1) - (6x^4 + 8x^3 - 2x - 3)$
2. Let $p(x) = 3x^4 + 7x^2 - 10x + 4$. Find $p(1)$, $p(0)$ and $p(-2)$.
3. Carry out of the following divisions and write your answer in the form $p(x) = f(x)q(x) + r(x)$.
   (a) $(3x^3 - x^2 + 4x + 7) \div (x + 2)$
   (b) $(3x^3 - x^2 + 4x + 7) \div (x^2 + 2)$
   (c) $(x^4 - 3x^2 - 2x + 4) \div (x - 1)$
   (d) $(5x^4 + 30x^3 - 6x^2 + 8x) \div (x^2 - 3x + 1)$
   (e) $(3x^4 + x) \div (x^2 + 4x)$
4. Find the quotient and remainder of the following divisions.

(a) $(2x^4 - 2x^2 - 1) \div (2x^3 - x - 1)$

(b) $(x^3 + 2x^2 - 5x - 3) \div (x+1)(x-2)$

(c) $(5x^4 - 3x^2 + 2x + 1) \div (x^2 - 2)$

(d) $(x^4 - x^2 - x) \div (x+2)^2$

(e) $(x^4 + 1) \div (x + 1)$

Section 3 Remainder Theorem

We have seein in Section 2 that if a polynomial $p(x)$ is divided by polynomial $f(x)$, where $\deg p(x) \geq \deg f(x)$, we obtain the expression $p(x) = f(x)q(x) + r(x)$, where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg r(x) < \deg f(x)$, or $r = 0$.

Now suppose $f(x) = x - a$, where $a \in \mathbb{R}$, then

$$\begin{aligned} p(x) &= (x-a)q(x) + r(x) \\ \text{i.e} \quad p(x) &= (x-a)q(x) + r, \quad \text{since } \deg r < \deg f. \end{aligned}$$

If we let $x = a$ then we get

$$\begin{aligned} p(a) &= (a-a)q(a) + r \\ \text{i.e.} \quad p(a) &= r. \end{aligned}$$

So the remainder when $p(x)$ is divided by $x - a$ is $p(a)$. This important result is known as the remainder theorem

<u>Remainder Theorem</u>: If a polynomial $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$.

<u>Example 1</u> :Find the remainder when $x^3 - 7x^2 + 4$ is divided by $x - 1$.

Instead of going through the long division process to find the remainder, we can now use the remainder theorem. The remainder when $p(x) = x^3 - 7x^2 + 4$ is divided by $x - 1$ is

$$p(1) = (1)^3 - 7(1)^2 + 4 = -2.$$

<u>Note</u>: Checking this using long division will give the same remainder of -2 (see Example 5 from Section 2).

Exercises:

1. Using the remainder theorem find the remainder of the following divisions and then check your answers by long divsion.

   (a) $(4x^3 - x^2 + 2x + 1) \div (x - 5)$

   (b) $(3x^2 + 12x + 1) \div (x - 1)$

2. Using the remainder theorem find the remainder of the following divisions.

   (a) $(x^3 - 5x + 6) \div (x - 3)$

   (b) $(3x^4 - 5x^2 - 20x - 8) \div (x + 1)$

   (c) $(x^4 - 7x^3 + x^2 - x - 1) \div (x + 2)$

   (d) $(2x^3 - 2x^2 + 3x - 2) \div (x - 2)$

Section 4 The Factor Theorem and Roots of Polynomials

The remainder theorem told us that if $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$. Notice theat the remainder $p(a) = 0$ then $(x - a)$ fully divides into $p(x)$ i.e. $(x - a)$ is a factor of $p(x)$. This is the factor theorem.

Factor Theorem: Suppose $p(x)$ is a polynomial and $p(a) = 0$. Then $(x - a)$ is a factor of $p(x)$ and we can write $p(x) = (x - a)q(x)$ for some polynomial $q(x)$.

Note: If $p(a) = 0$ we call $x = a$ a root of $p(x)$.

We can use trial and error to find solutions of polynomial $p(x)$ by finding a number a whre $p(a) = 0$. If we can find such a number a then we know $(x - a)$ is a factor of $p(x)$, and then we can use long division to find the remaining factors of $p(x)$.

Example 1 :

a) Find all the factors of $p(x) = 6x^3 - 17x^2 + 11x - 2$.
b) Hence find all the solutions to $6x^3 - 17x^2 + 11x - 2 = 0$.

Solution a) By trial and error notice that

$p(2) = 48 - 66 + 22 - 2 = 0$
i.e. 2 is a root of $p(x)$.
So $x - 2$ is a factor of $p(x)$.

To find other factors we'll divide $p(x)$ by $(x-2)$.

$$\begin{array}{r|l} & 6x^2 - 5x + 1 \\ \hline x-2 & 6x^3 - 17x^2 + 11x - 2 \\ & \underline{6x^3 - 12x^2} \\ & \quad\quad -5x^2 + 11x \\ & \quad\quad \underline{-5x^2 + 10x} \\ & \quad\quad\quad\quad\quad\quad x - 2 \\ & \quad\quad\quad\quad\quad\quad \underline{x - 2} \\ & \quad\quad\quad\quad\quad\quad\quad\quad 0 \end{array}$$

So $p(x) = (x-2)(6x^2 - 5x + 1)$. Now notice

$$\begin{aligned} 6x^2 - 5x + 1 &= 6x^2 - 3x - 2x + 1 \\ &= 3x(2x-1) - (2x-1) \\ &= (3x-1)(2x-1) \end{aligned}$$

So $p(x) = (x-2)(3x-1)(2x-1)$ and its factors are $(x-2)$, $(3x-1)$ and $(2x-1)$.

Solution b) The solutions to $p(x) = 0$ occur when

$$x - 2 = 0, \qquad 3x - 1 = 0, \qquad 2x - 1 = 0.$$

That is,

$$x = 2, \qquad x = \frac{1}{3}, \qquad x = \frac{1}{2}.$$

Exercises:

1. For each of the following polynomials find (i) its factors; (ii) its roots.
   (a) $x^3 - 3x^2 + 5x - 6$
   (b) $x^3 + 3x^2 - 9x + 5$
   (c) $6x^3 - x^2 - 2x$
   (d) $4x^3 - 7x^2 - 14x - 3$
2. Given that $x - 2$ is a factor of the polynomial $x^3 - kx^2 - 24x + 28$, find k and the roots of this polynomial.
3. Find the quadratic whose roots are -1 and $\frac{1}{3}$ and whose value at $x = 2$ is 10.
4. Find the polynomial of degree 3 which has a root at -1, a double root at 3 and whose value at $x = 2$ is 12.
5. (a) Explain why the polynomial $p(x) = 3x^2 + 11x^2 + 8x - 4$ has at least 1 root in the interval from $x = 0$ to $x = 1$.
   (b) Find all the roots of this polynomial.

Exercises for Worksheet 4.7

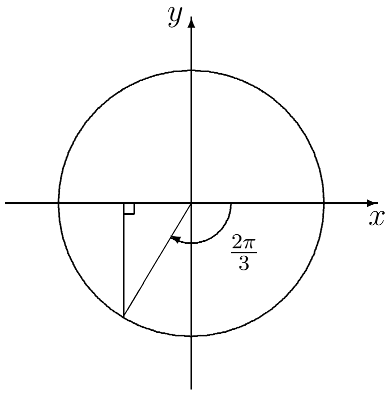
1. Find the quotient and remainder of the following divisions.
   (a) $(x^3 - x^2 + 8x - 5) \div (x^2 - 7)$
   (b) $(x^3 + 5x^2 + 15) \div (x + 3)$
   (c) $(2x^3 - 6x^2 - x + 6) \div (x - 6)$
   (d) $(x^4 + 3x^3 - x^2 - 2x - 7) \div (x^2 + 3x + 1)$
2. Find the factors of the following polynomials.
   (a) $3x^3 - 8x^2 - 5x + 6$
   (b) $x^3 - 4x^2 + 6x - 2$
   (c) $2x^3 + 5x^2 - 3x$
   (d) $x^3 + 6x^2 + 12x + 8$
3. Solve the following equations.
   (a) $x^3 - 3x^2 + x + 2 = 0$
   (b) $5x^3 + 23x^2 + 10x - 8 = 0$
   (c) $x^3 - 8x^2 + 21x - 18 = 0$
   (d) $x^3 - 2x^2 + 5x - 4 = 0$
   (e) $x^3 + 5x^2 - 4x + 20 = 0$
4. Consider the polynomial $p(x) = x^3 - 4x^2 + ax - 3$.
   (a) Find a if, when $p(x)$ is divided by $x + 1$, the remainder is -12.
   (b) Find all the factors of $p(x)$.
5. Consider the polynomial $h(x) = 3x^3 - kx^2 - 6x + 8$.
   (a) Given that $x - 4$ is a factor of $h(x)$, find k and find the other factors of $h(x)$.
   (b) Hence find all the roots of $h(x)$.
6. Find the quadratic whose roots are -3 and $\frac{1}{5}$ and whose value at $x = 0$ is -3.
7. Find the quadratic which has a remainder of -6 when divided by $x - 1$, a remainder of -4 when divided by $x - 3$ and no remainder when divided by $x + 1$.
8. Find the polynomial of degree 3 which has roots at $x = 1$, $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$, and whose value at $x = 2$ is -2.

Worksheet 4.8 Properties of Trigonometric Functions

Section 1 Review of Trigonometry

This section reviews some of the material covered in Worksheets 2.8, 3.3 and 3.4. The reader should be familiar with the trig ratios, using radians and working with exact values which arise from the following standard triangles.

Example 1 :Find the exact value of $\tan \frac{-2\pi}{3}$.



- $\frac{-2\pi}{3}$ lies in the third quadrant and the angle made with the horizontal axis is $\frac{\pi}{3}$
- tan is positive in the 3rd quadrant
- looking at the corresponding standard triangle in the third quadrant we see that $\tan(\frac{-2\pi}{3}) = +\sqrt{3}$

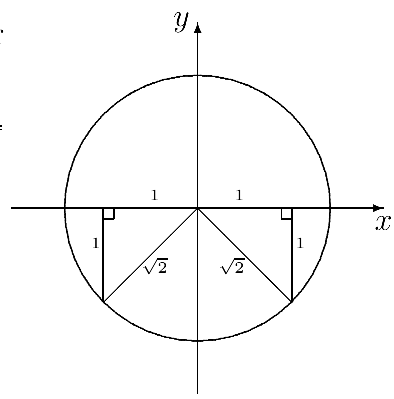
Example 2 :Find θ if $\sin\theta = -\frac{1}{\sqrt{2}}$ and $0 \le \theta \le 2\pi$.

- since $\sin\theta$ is negative it must lie in the third or fourth quadrant
- looking at the standard triangle where $\sin\theta = \frac{1}{\sqrt{2}}$ in the third and fourth quadrant we see that

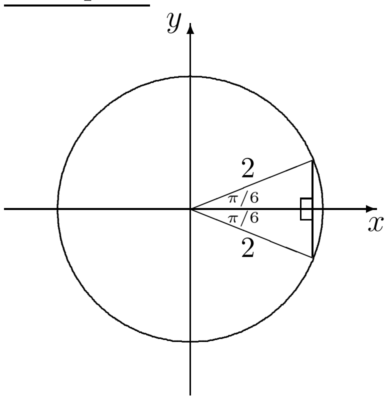
$$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

or

$$\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$



Example 3 :Find θ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $-\pi \leq \theta \leq \pi$.



- since $\cos\theta$ is positive it must lie in the first or fourth quadrant
- look at the standard triangles where $\cos\theta = \frac{\sqrt{3}}{2}$ in the first and fourth quadrant
- note that $-\pi \leq \theta \leq \pi$
- so $\theta = \frac{\pi}{6}$ or $\theta = -\frac{\pi}{6}$

Exercises:

1. Find the exact values of the following trig ratios.

   (a) $\tan(\frac{5\pi}{3})$
   (b) $\sin(\frac{-10\pi}{3})$
   (c) $\cos(\frac{9\pi}{4})$
   (d) $\sin(\frac{34\pi}{6})$
   (e) $\sec(\frac{5\pi}{6})$
   (f) $\cot(\frac{-11\pi}{4})$

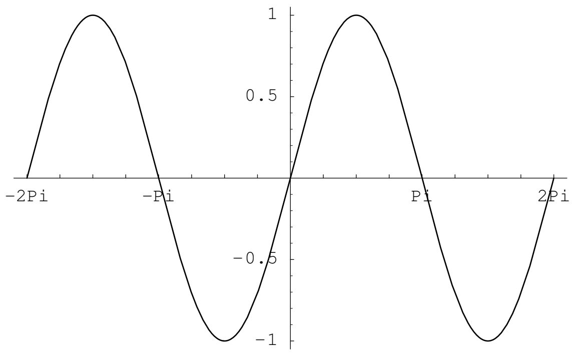
2. Find the value of θ in the following exercises.

   (a) $\cos\theta = -\frac{\sqrt{3}}{2}$ where $0 \leq \theta \leq 2\pi$
   (b) $\tan\theta = \frac{1}{\sqrt{3}}$ where $0 \leq \theta \leq 2\pi$
   (c) $\sin\theta = -\frac{\sqrt{3}}{2}$ where $-\pi \leq \theta \leq \pi$
   (d) $\sec\theta = -\sqrt{2}$ where $0 \leq \theta \leq 4\pi$
   (e) $\csc\theta = 2$ where $\frac{\pi}{2} \leq \theta \leq 2\pi$
   (f) $\tan^2\theta = 1$ where $0 \leq \theta \leq 2\pi$

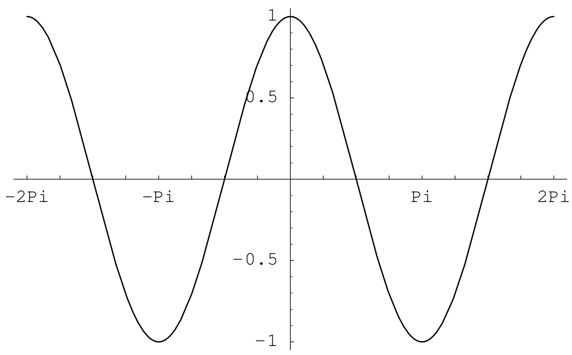
Section 2 Graphs of Trigonometric Functions

Recall the graphs of the trig functions described below for $-2\pi \leq x \leq 2\pi$.

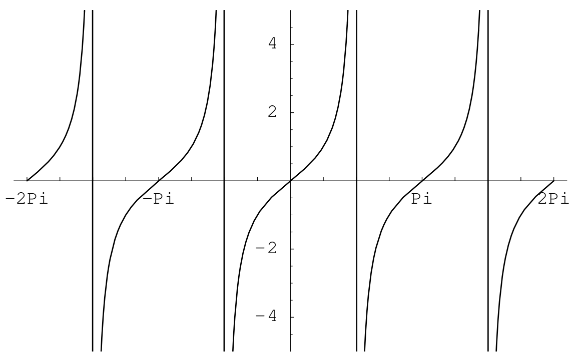
$y = \sin x$



$y = \cos x$



$y = \tan x$



From these graphs we can see some properties of these trig functions.

① These trig functions are periodic- they repeat themselves after a certain period.

- $\sin x$ and $\cos x$ have period 2π:

$$\text{i.e. } \begin{aligned} \sin x &= \sin(x+2\pi) \quad \forall x \in \mathbb{R} \\ \cos x &= \cos(x+2\pi) \quad \forall x \in \mathbb{R} \end{aligned}$$

- $\tan(x)$ has period π:

$$\text{i.e. } \tan(x) = \tan(x+\pi) \quad \forall x \in \mathbb{R}$$

② Note that $\sin x$ and $\cos x$ both lie between -1 and 1.

③ Note that $\tan x$ is undefined for $x = \frac{(2k-1)\pi}{2}, \quad k \in \mathbb{Z}$.

④ From the graphs we can see that

$$\begin{aligned} \sin\left(x+\frac{\pi}{2}\right) &= \cos x \\ \cos\left(\frac{\pi}{2}-x\right) &= \sin x \end{aligned}$$

⑤ Since $\sin x$ and $\tan x$ are odd functions we have

$$\begin{aligned} \sin(-x) &= -\sin x \quad \forall x \in \mathbb{R} \\ \tan(-x) &= -\tan x \quad \forall x \in \mathbb{R} \end{aligned}$$

Since $\cos x$ is an even function we have

$$\cos(-x) = \cos x \quad \forall x \in \mathbb{R}$$

These graphs alter if we change the the period or amplitude, or if there is a phase shift. Consider the graph of $y = \sin x$. In general we can think of this as $y = A\sin n(x-a)$, where

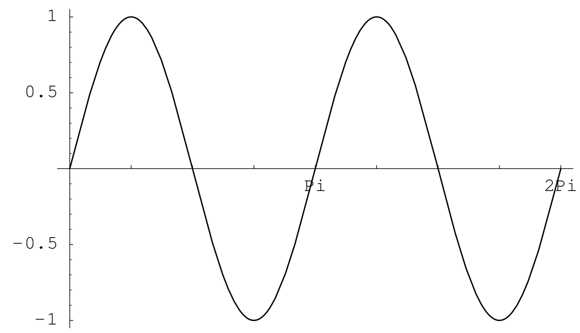
- A is the amplitude
- n alters the period (period$=\frac{2\pi}{n}$)
- by subtracting a from x, the graph shifts to the right by a.

So $y = \sin x$ has amplitude 1 and period 2π.

Example 1 :Sketch $y = \sin 2x$ for $0 \leq x \leq 2\pi$.

Here the period is now $\frac{2\pi}{2} = \pi$.

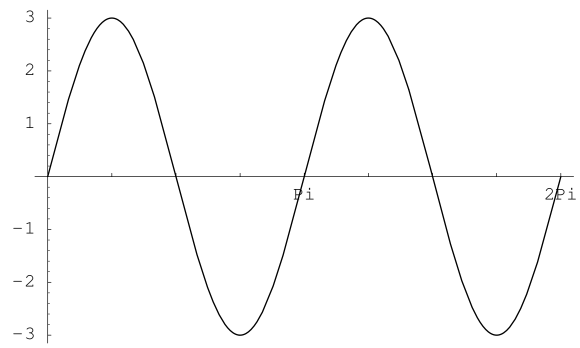
$\mathbf{y = \sin 2x}$



Example 2 :Sketch $y = 3\sin 2x$ for $0 \leq x \leq 2\pi$.

Here the period is still π but the amplitude is now 3.

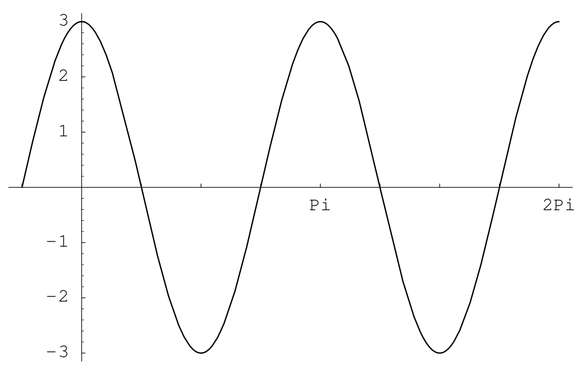
$\mathbf{y = 3\sin 2x}$



Example 3 :Sketch $y = 3\sin 2\left(x + \frac{\pi}{4}\right)$ for $-\frac{\pi}{4} \leq x \leq 2\pi$.

The period is π, the amplitude is 3 and there is a phase shift. The graph shifts to the left by $\frac{\pi}{4}$.

$\mathbf{y = 3\sin 2\left(x + \frac{\pi}{4}\right)}$



Exercises:

1. Sketch the following graphs stating the period for each.

   (a) $y = \cos 4x$

   (b) $y = 2\cos 4\left(x - \frac{\pi}{3}\right)$

   (c) $y = \tan 2\left(x + \frac{\pi}{2}\right)$

   (d) $y = 1 + \sin\left(\frac{x-\pi}{3}\right)$

   (e) $y = 2 - \cos\left(x + \frac{\pi}{6}\right)$

   (f) $y = |\cos x|$

   (g) $y = \cos|x|$

2. Solve the following equations for $0 \le x \le 2\pi$.

   (a) $3\cos^2 x - \cos x = 0$

   (b) $2\sin^2 x + \sin x - 1 = 0$

   (c) $4\cos^3 x - 4\cos^2 x - 3\cos x + 3 = 0$

   (d) $\tan^2 x + 2\tan x + 1 = 0$

Section 3 Trigonometric Identities

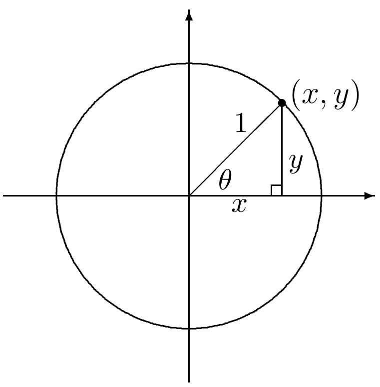
This section states and proves some common trig identities.

Pythagorean Identities

① $\cos^2\theta + \sin^2\theta = 1$

② $1 + \tan^2\theta = \sec^2\theta$

③ $1 + \cot^2\theta = \csc^2\theta$ Proof of ①: Consider a circle of radius 1 centred at the origin.



- Let θ be the angle measured anticlockwise for the positive x-axis.
- Using trig ratios we see that $x = \cos\theta$ and $y = \sin\theta$.
- By Pythagoras' Theorem $x^2 + y^2 = 1$ i.e. $\cos^2\theta + \sin^2\theta = 1$. □

Proof of ②: Divide both sides of identity ① by $\cos^2\theta$ and the result follows. □

Proof of ③: Divide both sides of identity ① by $\sin^2\theta$ and the result follows. □

Sum and Difference Identities

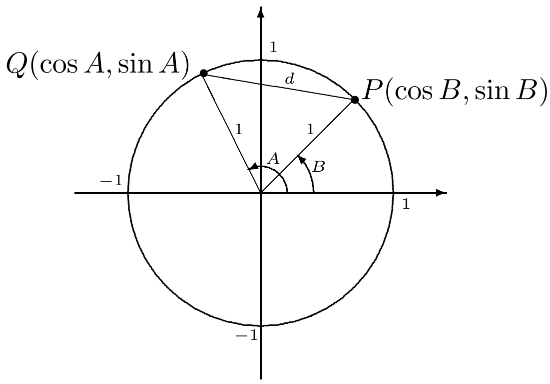
① $\sin(A+B) = \sin A\cos B + \cos A\sin B$

② $\sin(A-B) = \sin A\cos B - \cos A\sin B$

③ $\cos(A+B) = \cos A\cos B - \sin A\sin B$

④ $\cos(A-B) = \cos A\cos B + \sin A\sin B$

Proof of ① - ④: Consider the following circle of radius 1 with angles A and B as shown.



Note that we can label point P as $(\cos B\,,\,\sin B)$ and point Q as $(\cos A\,,\,\sin A)$ by using trig ratios. We can calculate the distance d using two methods.

Using the distance formula we see that

$$\begin{aligned} d^2 &= (\cos B - \cos A)^2 + (\sin B - \sin A)^2 \\ &= \cos^2 B - 2\cos A\cos B + \cos^2 A + \sin^2 B - 2\sin A\sin B + \sin^2 A \\ &= (\cos^2 B + \sin^2 B) + (\cos^2 A + \sin^2 A) - 2(\cos A\cos B + \sin A\sin B) \\ &= 2 - 2(\cos A\cos B + \sin A\sin B) \end{aligned}$$

Using the cosine rule we have

$$\begin{aligned} d^2 &= 1^2 + 1^2 - 2(1)(1)\cos(A-B) \\ &= 2 - 2\cos(A-B) \end{aligned}$$

Equating these we see that

$$\cos(A-B) = \cos A\cos B + \sin A\sin B,$$

thus proving sum and difference identity ④. □

We can use identity ④ to deduce the remaining identities. We have

$$\begin{aligned}\cos(A+B) &= \cos\left(A-(-B)\right)\\ &= \cos A\cos(-B)+\sin(A)\sin(-B)\\ &= \cos A\cos B-\sin A\sin B\end{aligned}$$

$$\begin{aligned}\sin(A+B) &= \cos\left(\frac{\pi}{2}-(A+B)\right)\\ &= \cos\left(\frac{\pi}{2}-A-B\right)\\ &= \cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B\\ &= \sin A\cos B+\cos\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-A\right)\right)\sin B\\ &= \sin A\cos B+\cos A\sin B\end{aligned}$$

$$\begin{aligned}\sin(A-B) &= \sin\left(A+(-B)\right)\\ &= \sin A\cos(-B)+\cos(A)\sin(-B)\\ &= \sin A\cos B-\cos A\sin B\end{aligned}$$

We have now established identities ① - ③. □

Double Angle Identities

① $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

②: $\sin 2\theta = 2\sin\theta\cos\theta$

Proof of ① and ②: using the sum and difference identities we can prove the double angle identities. For instance,

$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &= \cos\theta\cos\theta-\sin\theta\sin\theta\\ &= \cos^2\theta-\sin^2\theta.\end{aligned}$$

Replacing $\cos^2\theta$ by $1-\sin^2\theta$ (Pythagoeran identity ①) we can see that $\cos 2\theta = 1-2\sin^2\theta$.

Replacing $\sin^2\theta$ by $1-\cos^2\theta$ (Pythagoeran identity ①) we can see that $\cos 2\theta = 2\cos^2\theta-1$.

We also have

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta.\end{aligned}$$

We have now established identities ① and ②. □

<u>Half Angle Identities</u>

①: $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$

②: $\sin^2\left(\frac{\theta}{2}\right) = \frac{1 \cos\theta}{2}$

To prove the half angle identities we begin by rearranging the double angle identities.

Proof of ①: We take the double angle identity $\cos 2\theta = 2\cos^2\theta - 1$ to obtain

$$\begin{aligned}2\cos^2\theta &= \cos 2\theta + 1\\ \text{i.e. } \cos^2\theta &= \frac{\cos 2\theta + 1}{2}\\ \text{i.e. } \cos^2\left(\frac{\theta}{2}\right) &= \frac{\cos\theta + 1}{2}\end{aligned}$$

Proof of ②: We take the double angle identity $\cos 2\theta = 1 - 2\sin^2\theta$ to obtain

$$\begin{aligned}2\sin^2\theta &= 1 - \cos 2\theta\\ \text{i.e. } \sin^2\theta &= \frac{1-\cos 2\theta}{2}\\ \text{i.e. } \sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2}\end{aligned}$$

We have now established identities ① and ②. □

<u>Example 1</u> :Simplify $2\cos x\cos 2x\sin 3x - 2\sin x\sin 2x\sin 3x$.

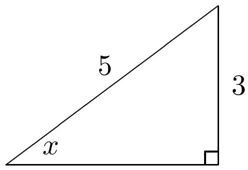
$$\begin{aligned}&2\cos x\cos 2x\sin 3x - 2\sin x\sin 2x\sin 3x\\ &= 2\sin 3x\,(\cos x\cos 2x - \sin x\sin 2x) && \text{(factorise)}\\ &= 2\sin 3x\,(\cos(x+2x)) && \text{(difference identity)}\\ &= 2\sin 3x\cos 3x\\ &= \sin 2(3x) && \text{(double angle)}\\ &= \sin 6x\end{aligned}$$

Example 2 :Show that $2\cos^2 2x - \cos^2 x - \sin^2 x = \cos 4x$.

$$\begin{aligned} 2\cos^2 2x - \cos^2 x - \sin^2 x &= 2\cos^2 2x - (\cos^2 x + sin^2 x) \\ &= 2\cos^2 2x - 1 \qquad \text{(Pythagorean identity)} \\ &= \cos 2(2x) \qquad \text{(double angle)} \\ &= \cos 4x \end{aligned}$$

Example 3 :Given $\sin x = \frac{3}{5}$ for $\frac{\pi}{2} \leq x \leq \pi$, find (i) $\cos x$ and (ii) $\sin 2x$.

Since $\frac{\pi}{2} \leq x \leq \pi$, we know x lies in the second quadrant. Also, since $\sin x = \frac{3}{5}$, we can form the following right angled triangle.



Using Pythagoras we see that the horizontal side is 4. We must have $\cos x = -\frac{4}{5}$, since cosine is negative in the second quadrant. So

$$\begin{aligned} \sin 2x &= 2\sin x \cos x \\ &= 2\left(\tfrac{3}{5}\right)\left(-\tfrac{4}{5}\right) \\ &= \tfrac{-4}{25} \end{aligned}$$

Example 4 :Use the appropriate half angle identity to find the exact value of $\sin\left(\frac{5\pi}{8}\right)$.

The half angle identity for sine is

$$\begin{aligned} \sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2} \\ \text{i.e. } \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \end{aligned}$$

Now,

$$\begin{aligned} \sin\left(\frac{\pi}{8}\right) &= \sin\left(\frac{\pi/4}{2}\right) \\ &= \pm\sqrt{\frac{1-\cos(\pi/4)}{2}} \\ &= \pm\sqrt{\frac{1-1/\sqrt{2}}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \\
&= \pm\sqrt{\frac{2-\sqrt{2}}{4}} \\
&= \pm\frac{\sqrt{2-\sqrt{2}}}{2}
\end{aligned}
$$

Since $\frac{\pi}{8}$ lies in the first quadrant, where sine is positive, we must have

$$\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Exercises:

1. Simplify the following.
   (a) $\cos 5x \sin x - \cos x \sin 5x$
   (b) $\dfrac{\sin^2 x + \cos^2 x + \tan^2 x}{\sec^2 x}$
   (c) $3 - 3\sin^2(x/8)$
   (d) $\dfrac{3\cot x \sin x \cos x}{\cos^2 x - \sin^2 x}$
   (e) $2\sin x \cos x - 4\sin^3 x \cos x$
   (f) $\dfrac{1}{2}\big(\cos(x/2) + 2\sin^2(x/4) - 1\big)$
2. Use the addition formulas to find the exact value of the following.
   (a) $\cos\left(\dfrac{7\pi}{12}\right)$ (b) $\sin\left(\dfrac{14\pi}{12}\right)$ (c) $\tan\left(\dfrac{7\pi}{6}\right)$
3. Use the half angle identities to calculate the exact value of the following.
   (a) $\cos\left(\dfrac{\pi}{8}\right)$ (b) $\cos\left(\dfrac{\pi}{12}\right)$ (c) $\sin\left(-\dfrac{\pi}{12}\right)$
4. Given $\tan x = \frac{5}{12}$ for $0 \le x \le \frac{\pi}{2}$, evaluate the following.
   (a) $\sin x$ (b) $\cos x$ (c) $\sin 2x$ (d) $\cos 2x$ (e) $\cos 3x$

Exercises for Worksheet 4.8

1. Solve the following equations.

   (a) $\tan x = -\sqrt{3}, \quad -\pi \leq x \leq \pi$

   (b) $\sin 2x = \frac{1}{2}, \quad 0 \leq x \leq 2\pi$

   (c) $4\sin^3 x - 3\sin x = 0, \quad 0 \leq x \leq 2\pi$

   (d) $\sec^2 x - 3\sec x + 2 = 0, \quad 0 \leq x \leq 2\pi$

2. Sketch the following curves.

   (a) $y = -2\cos\left(\frac{x}{3}\right)$

   (b) $y = 1 + \sin\left(x + \frac{\pi}{3}\right)$

   (c) $y = \tan 3\left(x - \frac{\pi}{6}\right)$

3. Prove the following.

   (a) $\cos 2x \sin x - \cos x \sin 2x = -\sin x$

   (b) $\dfrac{2\tan x - 2\sin^2 x \tan x}{\sin 2x} = 1$

   (c) $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$

   (d) $\dfrac{\cos 2x \cos 3x \cos 4x + \sin 2x \sin 3x \sin 4x}{2\sin x \sin 2x \cos x} = \cos x$

4. Suppose α lies in the third quadrant, β lies in the fourth quadrant and $\sin\alpha = -\frac{4}{5}$ with $\cos\beta = \frac{12}{15}$. Find the following.

   (a) $\sin(\alpha + \beta)$

   (b) $\cos(\alpha + \beta)$

   (c) $\tan(\alpha + \beta)$

5. Suppose $\cos x = \frac{2}{3}$ where $\frac{3\pi}{2} \leq x \leq 2\pi$. Find the exact value of the following.

   (a) $\cos 3x$

   (b) $\sin 3x$

6. Use the half angle identity to find the exact value of $\sin(-\frac{3\pi}{8})$.

7. Use the appropriate identities to find the exact values of the following.

   (a) $\cos\left(\frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right)$

   (b) $\sin\left(2\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)\right)$

Worksheet 4.9 Induction

Mathematical Induction is a method of proof. We use this method to prove certain propositions involving positive integers. Mathematical Induction is based on a property of the natural numbers, $\mathbb{N}$, called the Well Ordering Principle which states that evey nonempty subset of positive integers has a least element.

There are two steps in the method:

Step 1: Prove the statement is true at the starting point (usually $n = 1$).

Step 2: Assume the statement is true for n.
Prove the statement is true for $n + 1$ (using the assumption).

Example 5 :Prove $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$

Step 1: [We want to show this is true at the starting point $n = 1$.]

$$\begin{aligned} \text{LHS} &= 1 \\ \text{RHS} &= 1^2 = 1 \end{aligned}$$

Since LHS=RHS, the statement is true for $n = 1$.

Step 2: Assume the statement is true for n.
i.e. $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$
[Want to show this is true for $n + 1$.
i.e. Want to show $1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$]

$$\begin{aligned} \text{LHS} &= \underbrace{1 + 3 + 5 + \cdots + (2n - 1)}_{n^2} + (2n + 1) \\ &= n^2 + (2n + 1) \quad \text{(by assumption)} \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2 \\ &= \text{RHS} \end{aligned}$$

So, the statement is true for $n + 1$. Hence, the statement is true for all $n \in \mathbb{N}$, by induction. $\square$

Example 6 :Prove $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

Step 1: [We want to show this is true at the starting point $n = 1$.]

$$\begin{aligned} \text{LHS} &= \sum_{k=1}^{n} k^2 = 1^2 = 1 \\ \text{RHS} &= \tfrac{1}{6}1(1+1)\left(2(1)+1\right) = 1 \end{aligned}$$

Since LHS=RHS, the statement is true for $n = 1$.

Step 2: Assume the statement is true for n.

i.e. $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

[Want to show this is true for $n+1$.

i.e. Want to show $\sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$]

$$\begin{aligned} \text{LHS} &= \sum_{k=1}^{n+1} k^2 \\ &= \underbrace{1^2 + 2^2 + \cdots + n^2}+(n+1)^2 \\ &= \tfrac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{(by assumption)} \\ &= \tfrac{1}{6}(n+1)\left(n(2n+1) + 6(n+1)\right) \\ &= \tfrac{1}{6}(n+1)\left(2n^2 + 7n + 6\right) \\ &= \tfrac{1}{6}(n+1)(n+2)(2n+3) \\ &= \text{RHS} \end{aligned}$$

So, the statement is true for $n+1$. Hence, the statement is true for all $n \in \mathbb{N}$, by induction. □

Example 7 :Prove $2^n > n^2$ for $n > 5$.

Step 1: [We want to show this is true at the starting point $n = 5$.]

$$\begin{aligned} \text{LHS} &= 2^5 = 32 \\ \text{RHS} &= 5^2 = 25 \end{aligned}$$

Since LHS > RHS, the statement is true for $n = 5$.

Step 2: Assume the statement is true for n i.e. $2^n > n^2$.

[Want to show this is true for $n+1$ i.e. want to show $2^{n+1} > (n+1)^2$]

$$\begin{aligned}
\text{LHS} &= 2^{n+1} \\
&= 2^n \cdot 2 \\
&> 2n^2 \quad \text{(by assumption)} \\
&= n^2 + n^2 \\
&> n^2 + 2n + 1 \quad (\text{since } n^2 > 2n + 1 \text{ for } n \geq 5) \\
&= (n+1)^2 \\
&= \text{RHS}
\end{aligned}$$

So $2^{n+1} > (n+1)^2$ for $n \geq 5$ i.e. the statement is true for $n+1$ whenever $n \geq 5$. Hence, the statement is true for all $n \geq 5$, by induction. □

Example 8 :Prove that $9^n - 2^n$ is divisible by 7 for all $n \in \mathbb{N}$

Step 1: [We want to show this is true at the starting point $n = 1$.]

When $n = 1$, we have $9^1 - 7^1 = 7$ which is divisible by 7.

The statement is true for $n = 1$.

Step 2: Assume the statement is true for n.

i.e. Assume $9^n - 2^n$ is divisible by 7.

i.e. Assume $9^n - 2^n = 7m$ for some $m \in \mathbb{Z}$.

[Want to show this is true for $n+1$.

i.e. Want to show $9^{n+1} - 2^{n+1}$ is divisible by 7.]

$$\begin{aligned}
9^{n+1} - 2^{n+1} &= 9 \cdot 9^n - 2 \cdot 2^n \\
&= 9(7m + 2^n) - 2 \cdot 2^n \quad \text{(by assumption)} \\
&= 7(9m) + 9 \cdot 2^n - 2 \cdot 2^n \\
&= 7(9m) + 7 \cdot 2^n \\
&= 7(9m + 2^n),
\end{aligned}$$

which is divisible by 7. So the statement is true for $n+1$. Hence, the statement is true for all $n \in \mathbb{N}$, by induction. □

Exercises:

1. Prove the following propositions for all positive integers n.

   (a) $1 + 5 + 9 + 13 + \cdots + (4n - 3) = \frac{1}{2}n(4n - 2)$

   (b) $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

   (c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$

   (d) $10^1 + 10^2 + 10^3 + \cdots + 10^n = \frac{10}{9}(10^n - 1)$

   (e) $\sum_{r=1}^{n} r(r+1) = \dfrac{n(n+1)(n+2)}{3}$

   (f) $\sum_{k=1}^{n} \dfrac{1}{(3k-2)(3k-1)} = \dfrac{n}{3n+1}$

2. Prove the following by induction.

   (a) $2^n \geq 1 + n$ for $n \geq 1$

   (b) $3^n < (n+1)!$ for $n \geq 4$

3. Prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{N}$.

4. Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$.

5. Prove by induction that, if p is any real number satisfying $p > -1$, then $(1+p)^n \geq 1+np$ for all $n \in \mathbb{N}$.

6. Use induction to show that the nth derivative of x^{-1} is $\dfrac{(-1)^n n!}{x^{n+1}}$.

Answers to Test Four
and
Exercises from Worksheets 4.1 - 4.10

Answers to Test Four

Worksheet 4.1

Section 1

Section 2

Section 3

Section 4

Exercises 4.1

Worksheet 4.2

Section 1

Section 2

Section 3

Exercises 4.2

Worksheet 4.3

Exercises 4.3

Worksheet 4.4

Exercises 4.4

Worksheet 4.5

Exercises 4.5

Worksheet 4.6

Section 1

1. (a) $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4$
   (b) $\frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7}$
   (c) $3 + 5 + 7 + 9 + \cdots + (2n - 1)$
   (d) $2 + 4x + 6x^2 + 8x^3 + \cdots + 2^{n+1}x^n$
   (e) $1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \cdots + \frac{(-1)^n x^n}{2n+1}$

2. (a) $\displaystyle\sum_{n=1}^{6} n^2$
   (b) $\displaystyle\sum_{n=2}^{8} (-1)^n(2n-1)$
   (c) $\displaystyle\sum_{n=0}^{5} \frac{1}{3n+2}$
   (d) $\displaystyle\sum_{k=1}^{n} \frac{k+1}{k+2}$
   (e) $\displaystyle\sum_{k=1}^{2n+1} (-1)^{k+1}2^k$
   (f) $\displaystyle\sum_{n=1}^{15} 2nx^{2n+1}$
   (g) $\displaystyle\sum_{n=0}^{5} (-1)^n \frac{x^{n+1}}{n!}$
   (h) $\displaystyle\sum_{n=0}^{5} (4n+3)x^{n+1}$
   (i) $\displaystyle\sum_{k=3}^{n} (2k+2)x^{k+1}$
   (j) $\displaystyle\sum_{k=4}^{n} k(k-1)x^k$
   (k) $\displaystyle\sum_{n=1}^{99} \frac{(-1)^{n+1}x^{2n+1}}{n+1}$

Section 2

1. (a) $x^2 : \frac{3}{2}, \quad x^6 : \frac{7}{6!}$
   (b) $x^2 : 20, \quad x^6 : 72$
   (c) $x^2 : \frac{1}{3}, \quad x^6 : -\frac{1}{4}$
   (d) $x^2 : 13, \quad x^6 : 33$
   (e) $x^2 : 0, \quad x^6 : \frac{1}{5!} - \frac{1}{4!}$
   (f) $x^2 : 4, \quad x^6 : 20$

Exercises 4.6

1. (a) $-x + 4x^2 - 9x^3 + 16x^4 - 25x^5 + 36x^6 - 49x^7$
   (b) $\frac{2}{4}x^6 + \frac{3}{5}x^8 + \frac{4}{6}x^{10} + \frac{5}{7}x^{12} + \frac{6}{8}x^{14} + \frac{7}{9}x^{16}$
   (c) $2x^6 + 6x^9 + 12x^{12} + 20x^{15} + \cdots + n(n+1)x^{3n+3}$

2. (a) $\displaystyle\sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^2}$ (d) $\displaystyle\sum_{k=2}^{n} \frac{1}{(k+5)!}$ (g) $\displaystyle\sum_{k=1}^{10} (-1)^{k+1}(4k+1)x^k$

(b) $\displaystyle\sum_{n=0}^{8} (-1)^n(3n+7)$ (e) $\displaystyle\sum_{n=1}^{12} 3nx^{2n}$ (h) $\displaystyle\sum_{k=5}^{n} \frac{kx^{k-2}}{2}$

(c) $\displaystyle\sum_{k=2}^{n} (k+2)^2$ (f) $\displaystyle\sum_{n=0}^{12} \frac{x^{2n+7}}{n!}$ (i) $\displaystyle\sum_{k=6}^{n+1} nx^{2n}$

3. (a) $x : -\frac{1}{11!}, \quad x^3 : -\frac{1}{19!}, \quad x^7 : -\frac{1}{35!}$
(b) $x : \frac{3}{25}, \quad x^3 : \frac{27}{49}, \quad x^7 : \frac{3^7}{121}$
(c) $x : -1, \quad x^3 : \frac{4}{3}, \quad x^7 : \frac{31}{14}$
(d) $x : -\frac{1}{3}, \quad x^3 : -\frac{1}{2}, \quad x^7 : -\frac{1}{3}\left(\frac{1}{6!} + \frac{1}{5!}\right)$
(e) $\frac{15}{7!} + \frac{1}{3!}, \quad x^3 : \frac{25}{11!} + \frac{3}{7!} \quad x^7 : \frac{45}{19!} + \frac{7}{15!}$

4. (a) $-(n+1)^2$ (b) $-\dfrac{n}{n+1}$

Worksheet 4.7

Section 1

1. (a) (i) 1 (ii) 1 (iii) 4 (d) (i) 6 (ii) -3 (iii) 5
(b) (i) 9 (ii) 0 (iii) 3 (e) (i) 0 (ii) 7 (iii) 6
(c) (i) -2 (ii) 0 (iii) 1 (f) (i) 7 (ii) -6 (iii) 4

2. $a = 5$, $b = \pm 2$, $c = 0$, $d = 2$ and $e = -7$.

Section 2

1. (a) $-x^2 - 10x, \quad \deg = 2$
(b) $4x^5 + 12x^4 - 3x^3 - 10x^2 - 3x, \quad \deg = 5$
(c) $x^4 + 4x^3 + 6x^2 + 4x + 1, \quad \deg = 4$
(d) $-x^4 - 15x^3 + 6x + 4, \quad \deg = 4$

2. $p(1) = 4, \quad p(0) = 4, \quad p(-2) = 100$

3. (a) $3x^3 - x^2 + 4x + 7 = (x+2)(3x^2 - 7x + 18) - 29$

(b) $3x^3 - x^2 + 4x + 7 = (x^2 + 2)(3x - 1) + (-2x + 9)$

(c) $x^4 - 3x^2 + 2x + 4 = (x - 1)(x^3 + x^2 - 2x) + 4$

(d) $5x^4 + 30x^3 - 6x^2 + 8x = (x^2 - 3x + 1)(5x^2 + 45x + 64) + (155x - 64)$

(e) $3x^4 + x = (x^2 + 4x)(3x^2 - 12x + 48) - 191x$

4. (a) $q(x) = x, \quad r(x) = -x^2 + x - 1$

(b) $q(x) = x + 3, \quad r(x) = 3$

(c) $q(x) = 5x^2 + 7, \quad r(x) = 2x + 15$

(d) $q(x) = x^2 - 4x + 15, \quad r(x) = 75x + 60$

(e) $q(x) = x^3 - x^2 + x - 1, \quad r(x) = 2$

Section 3

1. (a) 486 (b) 16

2. (a) 18 (b) 10 (c) 77 (d) 12

Section 4

1. (a) $(x - 2)(x^2 - x + 3)$

(b) $(x - 1)^2(x + 5)$

(c) $x(2x + 1)(3x - 2)$

(d) $(4x + 1)(x - 3)(x + 1)$

2. $k = 3$, roots: 2 (double), -7

3. $6x^2 + 4x - 2$

4. $4x^3 - 20x^2 + 12x + 36$

5. (b) $\frac{1}{3}$, -2

Exercises 4.7

1. (a) $q(x) = x - 1, \quad r(x) = x + 2$

(b) $q(x) = x^2 - 2x - 6, \quad r = 33$

(c) $q(x) = 2x^2 - 1, \quad r = 0$

(d) $q(x) = x - 1, \quad r(x) = 4x - 5$

2. (a) $(3x - 2)(x + 1)(x - 3)$

(b) $(x - 2)(x^2 - 2x + 2)$

(c) $x(2x-1)(x+3)$

(d) $(x-1)(x^2+x-2)$

(e) $(x+2)^3$

3. (a) $2, \quad \frac{-1+\sqrt{5}}{2}, \quad \frac{-1-\sqrt{5}}{2}$

   (b) $\frac{2}{5}, \quad -1, \quad -4$

   (c) $3, \quad 2$

   (d) 1

   (e) $2, \quad -2, \quad -5$

4. (a) $a = 5$

   (b) factors are $(x-3)$ and (x^2-x+1)

5. (a) $k = 11$, factors are $(x-4)$, $(x+1)$ and $(3x-2)$

   (b) roots are 4, -1 and $\frac{2}{3}$

6. $5x^2 + 14x - 3$

7. $x^2 - 3x - 4$

8. $2x^3 - 6x^2 + 2x + 2$

Worksheet 4.8

Section 1

1. (a) $-\sqrt{3}$

   (b) $\dfrac{1}{\sqrt{2}}$

   (c) $-\dfrac{2}{\sqrt{3}}$

   (d) $\dfrac{\sqrt{3}}{2}$

   (e) $-\dfrac{\sqrt{3}}{2}$

   (f) 1

2. (a) $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$

   (b) $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$

   (c) $-\dfrac{\pi}{3}, -\dfrac{2\pi}{3}$

   (d) $\dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{11\pi}{4}, \dfrac{13\pi}{4}$

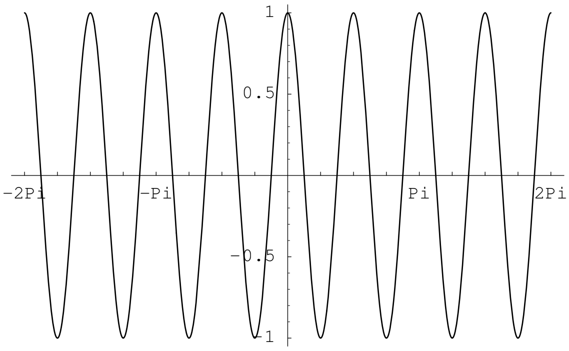
   (e) $\dfrac{5\pi}{6}$

   (f) $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Section 2

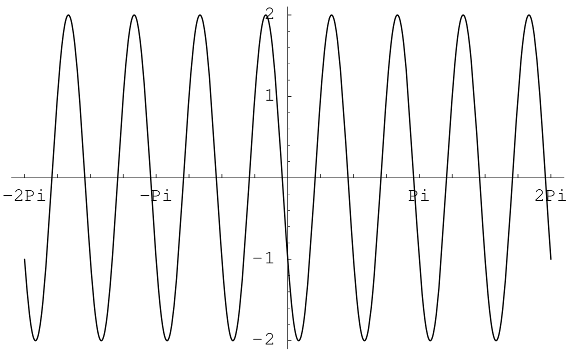
1. (a) period is $\frac{\pi}{2}$

$\mathbf{y = \cos 4x}$



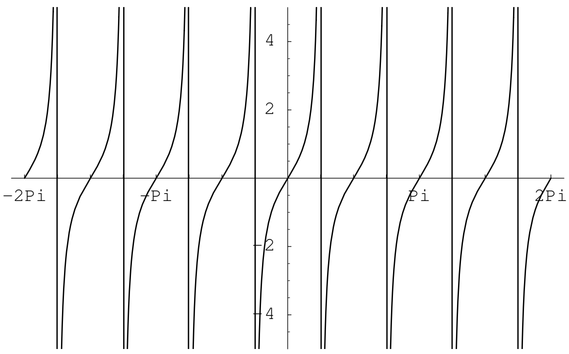
(b) period is $\frac{\pi}{2}$

$\mathbf{y = 2\cos 4\left(x - \frac{\pi}{3}\right)}$



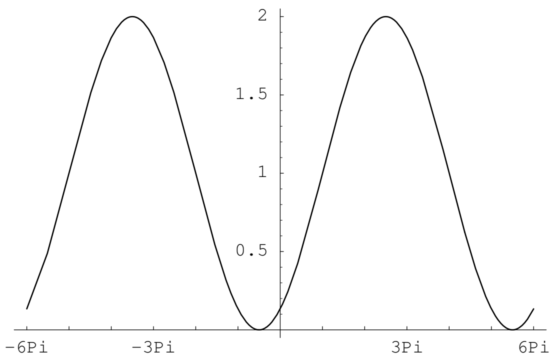
(c) period is $\frac{\pi}{2}$

$\mathbf{y = \tan 2\left(x + \frac{\pi}{2}\right)}$



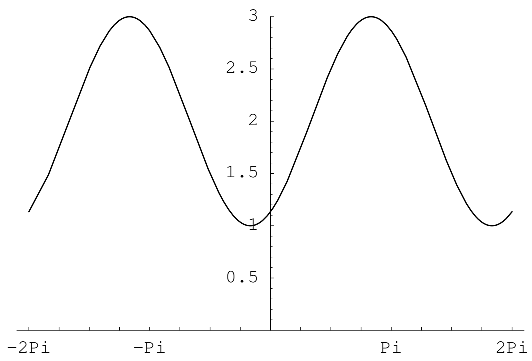
(d) period is 6π

$\mathbf{y = 1 + \sin\left(\frac{x - \pi}{3}\right)}$



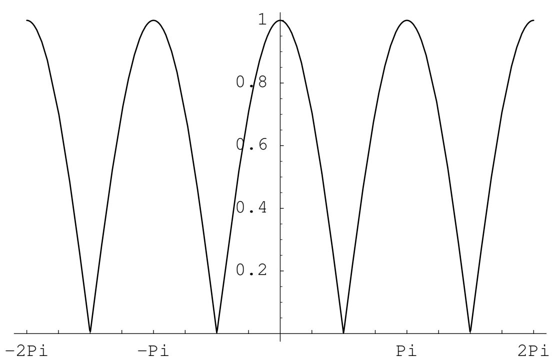
(e) period is 2π

$$\mathbf{y = 2 - \cos\left(x + \frac{\pi}{6}\right)}$$



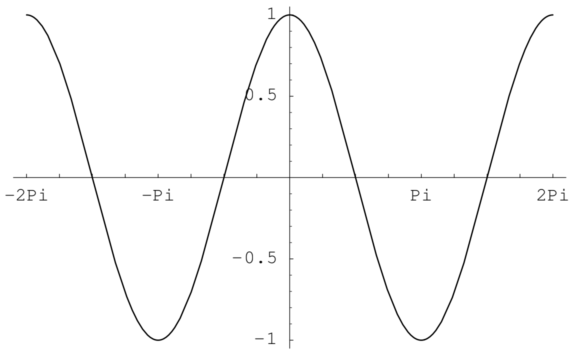
(f) period is π

$$\mathbf{y = |\cos x|}$$



(g) period is 2π

$$\mathbf{y = \cos |x|}$$



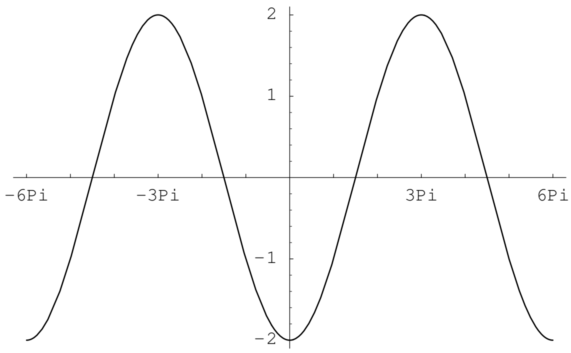
2. (a) $\frac{\pi}{2}, \frac{3\pi}{2}, 0.95, 5.34$
   (b) $\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$
   (c) $0, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
   (d) $\frac{\pi}{4}, \frac{5\pi}{4}$

Section 3

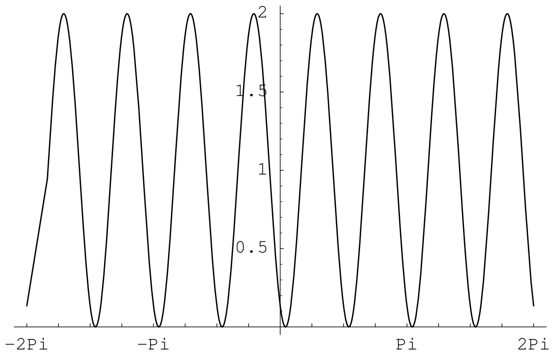
1. (a) $-\sin 4x$
   (b) 1
   (c) $3\cos\left(\frac{x}{4}\right)$
   (d) $\frac{3}{2}$
   (e) $\sin 4x$
   (f) 1
2. (a) $\frac{\sqrt{2}-\sqrt{6}}{4}$
   (b) $-\frac{1}{2}$
   (c) $\sqrt{3}$
3. (a) $\frac{\sqrt{2+\sqrt{2}}}{2}$
   (b) $\frac{\sqrt{2+\sqrt{3}}}{2}$
   (c) $-\frac{\sqrt{2-\sqrt{3}}}{2}$
4. (a) $\frac{5}{13}$
   (b) $\frac{12}{13}$
   (c) $\frac{120}{169}$
   (d) $\frac{119}{169}$
   (e) $\frac{276}{715}$

Exercises 4.8

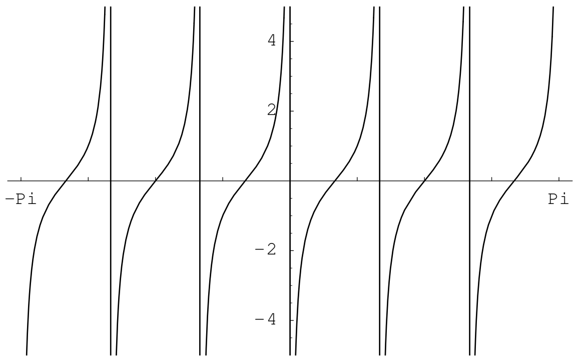
1. (a) $\frac{2\pi}{3}, -\frac{\pi}{3}$
   (b) $\frac{\pi}{12}, \frac{5\pi}{12}$
   (c) $0, \pi, 2\pi, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
   (d) $0, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$
2. (a) $\mathbf{y = -2\cos\left(\frac{x}{3}\right)}$



(b) $\mathbf{y = 1 + \sin 4\left(x + \frac{\pi}{3}\right)}$



(c) $\mathbf{y = \tan 3\left(x - \frac{\pi}{6}\right)}$



3. Proofs only.

4. (a) $-\dfrac{7}{25}$ (b) $-\dfrac{24}{75}$ (c) $\dfrac{7}{24}$

5. (a) $-\dfrac{119}{54}$ (b) $-\dfrac{61\sqrt{5}}{72}$

6. $-\dfrac{\sqrt{2+\sqrt{2}}}{2}$

7. (a) $\sqrt{\dfrac{4+\sqrt{2}+\sqrt{6}}{8}}$ (b) $-\dfrac{1}{2}$