Worksheet 4.5 Binomial Coefficients

Section 1 Introduction

We wish to be able to expand an expression of the form $(ax+b)^n$. We can do this easily for $n = 2$, but what about a large n? It would be tedious to manually multiply $(ax+b)$ by itself 10 times, say. There are two methods of expanding an expression of this type without doing all the multiplications involved. The first method we will look at is called Pascal's triangle. For a given n, this method uses the expansion of $n-1$. The first 5 rows of Pascal's triangle are shown:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Pascal's triangle is particularly useful when dealing with small n. The triangle is easy to remember how to reproduce as each entry is the sum of the two right and left entries on the line above, and the sides are always one. Thus for the second entry of line five we get

$$
\begin{array}{ccc}
1 & & 3 \\
 & \searrow\!\swarrow & \\
 & 4 &
\end{array}
$$

For the expansion of $(x+1)^2$, we use the third line of the triangle. For the expansion of $(x+1)^n$, we need the $(n+1)$th line of Pascal's triangle, and we have the following rule: the 1st term in the line is the coefficient of x^n, the second term in the line is the coefficient of x^{n-1}, and so on until the $(n+1)$th term in the line (the last term) is the coefficient of $1^n = 1$. If you wish to use Pascal's triangle on an expansion of the form $(ax+b)^n$, then some care is needed. The $(n+1)$th row is the row we need, and the 1st term in the row is the coefficient of $(ax)^n b^0$. The second term in the row is the coefficient of $(ax)^{n-1}b^1$. The last term in the row - the $(n+1)$th term - is the coefficient of $(ax)^0 b^n$. Care should be taken when minus signs are involved.

Example 1 : Expand $(1+x)^4$. Using the coefficients in the fifth row,

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

Example 2 : What is the coefficient of x^2 in $(x+2)^5$? The 6th line from Pascal's triangle is 1 5 10 10 5 1. The x^2 term is the 4th term in the expansion, so we pick up 10 from Pascal's triangle. But the expression we are raising to the power 5 is of the form $(ax+b)$ with $a=1$ and $b=2$. So the fourth term also has a factor a^2b^3 which in this case is $2^3=8$. So the coefficient of x^2 is $10\times 8=80$.

Example 3 : Expand the expression $(ax+b)^3$. The 4th row of Pascal's triangle is 1 3 3 1. So

$$\begin{aligned}(ax+b)^3 &= (ax)^3+3(ax)^2b+3(ax)^1b^2+b^3\\ &= a^3x^3+3a^2bx^2+3ab^2x+b^3\end{aligned}$$

Notice that the powers of a and b in each term always add to give the power of the expansion. This is always the case.

Example 4 : Find the constant term (the term that is independent of x) in the expansion of $(x-2)^5$. The constant term is the last term, and is $(-2)^5$. Notice that the minus sign is important.

Example 5 : Find the 4th term in the expansion of $(2x-3)^5$. The 4th term in the 6th line of Pascal's triangle is 10. So the 4th term is

$$10(2x)^2(-3)^3=-1080x^2$$

The 4th term is $-1080x^2$.

The second method to work out the expansion of an expression like $(ax+b)^n$ uses binomial coefficients. This method is more useful than Pascal's triangle when n is large. To work out binomial coefficients, we need to know what $n!$ - which is read as n factorial - means. It is defined by

$$n!=n\times(n-1)\times(n-2)\times\ldots\times 3\times 2\times 1$$

In addition, we define $0!=1$.

Example 6 : What is 4!?

$$4!=4\times 3\times 2\times 1=24$$

Example 7 : What is 5!?

$$5!=5\times 4!=5\times 4\times 3\times 2\times 1=120$$

Example 8 : Evaluate $\frac{7!}{5!}$.

$$\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$$

Example 9 : Evaluate $\frac{n!}{(n-3)!3!}$.

$$\begin{aligned} \frac{n!}{(n-3)!3!} &= \frac{n \times (n-1) \times \ldots \times 2 \times 1}{(n-3) \times (n-4) \times \ldots \times 2 \times 1 \times 3 \times 2 \times 1} \\ &= \frac{n \times (n-1) \times (n-2)}{6} \end{aligned}$$

The binomial coefficients can be used to find the expansion of $(ax+b)^n$ - the coefficient of x^k is given by

$$\frac{n!}{(n-k)!k!} a^k b^{n-k}$$

where $\frac{n!}{(n-k)!k!}$ is called the binomial coefficient.

Example 10 : What is the coefficient of x^{10} in the expansion of $(x+1)^{12}$?

$$\frac{12!}{(12-10)!10!} 1^{10} 1^2 = \frac{12!}{10!2!} = 66$$

The coefficient of x^{10} in $(x+1)^{12}$ is 66.

Example 11 : What is the coefficient of x in the expansion of $(2x-1)^8$?

$$\frac{8!}{(8-1)!1!} 2^1 (-1)^7 = -\frac{8!}{7!} = -16$$

Example 12 : What will be the coefficient of x^5 in $\frac{1}{x^2}(2x-3)^9$? This will be the same as the coefficient of x^7 in $(2x-3)^9$, which is given by

$$\frac{9!}{(9-7)!7!} \times 2^7 \times (-3)^{9-7} = \frac{9!}{2!7!} 2^7 (-3)^2 = 81 \times 8 \times 2^6 = 3^4 \times 2^9$$

The answer can be left like this or, if you prefer, you can calculate further.

Exercises 4.5 Binomial Coefficients

1. (a) Evaluate 9!

   (b) Evaluate $\dfrac{5!}{2!3!}$

   (c) Show that ${}^5C_2 = {}^5C_3$, where ${}^nC_k = \frac{n!}{k!(n-k)!}$.

   (d) Prove that ${}^nC_k = {}^nC_{n-k}$

   (e) Prove that $\dfrac{(2n)!}{2^n n!} = \dfrac{1}{2}^n \, {}^{2n}C_n n!$

   (f) Write down expansions of the following:

   i. $(2x+3y)^4$

   ii. $(a+\frac{1}{a})^6$

   iii. $(\frac{a}{b}-\frac{b}{a})^7$

   iv. $(x^2+a)^5$

2. (a) Write down and simplify the 4th term of

   i. $(\frac{m}{2}+3n)^8$

   ii. $(x-\frac{1}{x})^n$

   (b) Find the coefficients of

   i. x^2 in $(x+\frac{1}{x})^8$

   ii. a^5b^4 in $(3a-\frac{b}{3})^9$

   (c) Find the constant terms in

   i. $(2x-\frac{1}{x^2})^9$

   ii. $(2x+\frac{1}{x})^{2n}$

   (d) Find the largest positive numerical term in the expansion of $(1-2x)^9$ if $x=3$

3. (a) Express $(1.08)^3$ as a binomial of the form $(a+b)^n$, and evaluate it.

   (b) The number of ways of choosing k objects from n objects is given by ${}^nC_k$.

   i. How many different sets of three colours can be selected from the colours red, orange, yellow, green, blue, and violet?

   ii. In how many ways can a team of five basketball players be selected from 8 boys?

   iii. A secretary has nine letters and only five stamps. How many ways can he select the letters for posting?

   iv. In a plane, there are 5 points, no three of which are collinear. How many different triangles can be drawn by joining sets of three points?