Worksheet 1.2 Factorization of Integers

Section 1 Factors

When we multiply two numbers a and b together we get what is called the product of a and b. We call a and b the factors or divisors of ab. Many numbers have more than two factors, for instance:

We have $6 \times 2 = 12$ and $3 \times 4 = 12$. So 2 and 6 are factors of 12 and so are 3 and 4.

Factors come in pairs. When we wish to find all the factors of a number we do it systematically. We test to find all the numbers that divide evenly into the number we are trying to find the factors of.

Example 1 : Find the factors of 15.

1 clearly divides evenly into 15: $15 \times 1 = 15$. So 1 and 15 are factors of 15.

2 doesn't divide 15 evenly, so 2 is not a factor of 15.

3 goes evenly into 15 ($5 \times 3 = 15$) so 3 and 5 are factors of 15.

4 doesn't divide 15 evenly, so 4 is not a factor of 15.

5 was found to be a factor when we investigated 3. Since factors come in pairs, anything paired with a number bigger than 5 will be smaller than three, and we have already looked at these. When a number is found to be a factor, as 5 is, and its pair (in this case, 3) is smaller than the number you are looking at, all the factors have been found.

The factors of 15 are then: 1,3,5,15.

3 and 5 are called proper factors of 15. 1 and 15 are not proper factors since 1 divides everything evenly.

Example 2 : Find all the factors of 36.

$$\begin{aligned} 1 \times 36 &= 36 \\ 2 \times 18 &= 36 \\ 3 \times 12 &= 36 \\ 4 \times 9 &= 36 \\ 6 \times 6 &= 36 \end{aligned}$$

After 6×6 the answer will be smaller than the divisor (why?) so the factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36.

Any number can be represented by its factors; for instance if I had \$20 I could also say I had four lots of \$5. Factors are useful in at least two ways:

1. Simplifying complicated expressions.
2. Simplifying fractions.

Example 3 :

$$\frac{10}{15} = \frac{2 \times 5}{3 \times 5}$$

We can cancel the 5 from the top and the bottom to get

$$\frac{10}{15} = \frac{2}{3}$$

We can say that $\frac{10}{15}$ and $\frac{2}{3}$ are equivalent fractions.

There will be more about simplifying fractions in another worksheet.

Exercises:

1. Find the factors of each of the following numbers:
   (a) 18
   (b) 24
   (c) 56
   (d) 128

Section 2 Prime Factorization

There is a special group of numbers called primes. These are numbers which have no proper factors. In other words the only numbers that divide evenly into primes are 1 and themselves. The first few primes are 2, 3, 5, 7, 11 and 13. The list goes on indefinitely.

Note: Primes are numbers which have no proper factors.

There are three types of whole numbers:

1. Composites: this is the name we give to numbers which have proper factors.
2. Primes: numbers which do not have proper factors.
3. Units: 1 is a unit.

In the previous section we saw how to represent composite numbers by the products of their factors. 12 may be represented in several ways:

$$12 = 6 \times 2 = 3 \times 4 = 3 \times 2 \times 2$$

$3 \times 2 \times 2$ is called the prime factorization of 12 since all the factors are prime. The prime factorization of a number is the product of primes which represent the number. Note that the 2 appears twice; we need this to actually represent 12 otherwise we would have $2 \times 3 = 6$ which is not equal to 12. The prime factorization of a number is unique. This means that there is only one way of writing a number as a product of primes (if you don't count the order it's written in).

Example 1 : Find the prime factorization of

- 7 $\quad 7 = 7$ since 7 is a prime.
- 45 $\quad 45 = 9 \times 5 = 3 \times 3 \times 5$
- 32 $\quad 32 = 2 \times 16 = 2 \times 2 \times 8 = 2 \times 2 \times 2 \times 4 = 2 \times 2 \times 2 \times 2 \times 2$

So, how do we find the prime factorization of a number? This is similar to finding the factors of a number. Test to see if the smaller primes divide evenly into the number first.

Example 2 : Find the prime factorization of 180

Is 2 a factor of 180? Yes: $180 = 2 \times 90$.

Is 2 a factor of 90? Yes: $90 = 2 \times 45$ and $180 = 2 \times 2 \times 45$.

Is 2 a factor of 45? No.

Is 3 a factor of 45? Yes: $45 = 3 \times 15$. So $180 = 2 \times 2 \times 3 \times 15$.

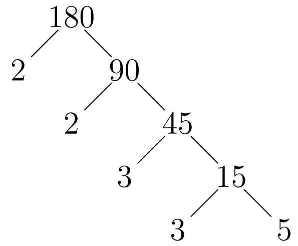
Is 3 a factor of 15? Yes: $15 = 3 \times 5$. So $180 = 2 \times 2 \times 3 \times 3 \times 5$.

5 is a prime, so we are finished.

$$180 = 2 \times 2 \times 3 \times 3 \times 5$$

Note: If we had tested 3 first we still would have ended up with the same factorization. It is important to make sure that you list all the primes the correct number of times because we use the prime factorization to represent the number.

Another way of finding the prime factorizations of a number is by using a factor tree. Here is a factor tree for 180.



When you use a prime number, stop and go to the other branch of the tree. So the prime factorization of 180 is

$$180 = 2 \times 2 \times 3 \times 3 \times 5$$

Exercises:

1. Find the prime factorization of each of the following numbers:

   (a) 84
   (b) 72
   (c) 126
   (d) 480
   (e) 26

Section 3 Highest Common Factors

A *common factor* of two numbers is a number which divides both of them (in other words a number which is a factor of both of them).

The *highest common factor* is the largest number which divides both numbers. The highest common factor is also called the *greatest common divisor*.

The factors of 12 are $1, 2, 3, 4, 6$ and 12. The factors of 18 are $1, 2, 3, 6, 9$ and 18. The highest common factor is the biggest number that appears in both lists. In this case, the answer is 6.

Note : To find the highest common factor of two numbers you can find all the factors of the smaller number; don't forget the number itself and, starting from the largest, test to see if they divide evenly into the bigger number.

Example 1 : Find the highest common factor of 72 and 180.

72 has factors $1, 2, 4, 6, 8, 9, 12, 18, 36$ and 72.

Does 72 go evenly into 180? No.

Does 36 go evenly into 180? Yes.

So 36 is the highest common factor of 72 and 180.

Example 2 : Find the highest common factor of 7 and 28.

7 has factors 1 and 7 only.

Does 7 go evenly into 28? Yes.

So 7 is the highest common factor of 7 and 28.

Example 3 : Find the highest common factor of 5 and 32.

5 is prime so its only factors are 1 and 5.

Does 5 go evenly into 32? No.

Clearly 1 will go evenly into 32, so 1 is the highest common factor of 5 and 32.

These last two examples illustrate what happens when one of the numbers is prime. If the larger number is prime the highest common factor will be 1 and if the smaller number is prime the highest common factor will either be 1 or the smaller number itself.

Exercises:

1. Find the highest common factor of these numbers:
   (a) 12 and 18
   (b) 15 and 45
   (c) 24 and 148
   (d) 56 and 242
   (e) 17 and 63

Section 4 Lowest Common Multiple

Given any two numbers we can find the smallest number which is a multiple of each of them; in other words we can find the smallest number which has both of them as factors. This number is called the lowest common multiple of the two numbers. The lowest common multiple is not always just the two numbers multiplied together.

Example 1 : Find the lowest common multiple of 9 and 15.

We are looking for a number which has 9 and 15 as factors.

Multiples of 9 are $9, 18, 27, 36, 45, \ldots$

Multiples of 15 are $15, 30, 45, 60, \ldots$

Notice that 45 is the smallest number which occurs in both lists, so 45 is the lowest common multiple of 9 and 15. It is not $9 \times 15 = 135$ as we might have expected.

You could write down lists to find the lowest common multiple as was done in example 1, but the lists might get very long before you get a number in common. An easier way is outlined below.

First we find the highest common factor of the two numbers as outlined in the last section. So for our example above the highest common factor of 9 and 15 is 3.

Now we multiply 3 by the uncommon factors, i.e. the missing partners of 3 from the factorization of the initial two numbers. From our example we have $9 = 3 \times 3$ and $15 = 3 \times 5$, so the missing partners are 3 and 5. When we multiply these together with the highest common factor we get the lowest common multiple. So $3 \times 3 \times 5 = 45$.

Lets see if we can see why this works.

$$\begin{array}{rclcrcl} 9 & = & 3 \times 3 & \text{and} & 15 & = & 3 \times 5 \\ 45 & = & \overbrace{3 \times 3}^{9} \times 5 & & 45 & = & 3 \times \overbrace{3 \times 5}^{15} \\ 45 & = & 9 \times 5 & & 45 & = & 3 \times 15 \end{array}$$

So the factors of both 9 and 15 appear in the lowest common multiple because of the way we set it up. They appear in the factorization only once because we only added the common factor once.

Example 2 : Find the lowest common multiple of 180 and 72.

The highest common factor of 72 and 180 is 36 (as we found from the previous section).

$$\begin{aligned}
72 &= 36 \times 2 \quad \text{and} \\
180 &= 36 \times 5 \\
\text{LCM} &= \overbrace{36 \times 2}^{72} \times 5 \\
\text{LCM} &= 2 \times \overbrace{36 \times 5}^{180} \\
360 &= 36 \times 2 \times 5 \\
360 &= 36 \times 5 \times 2
\end{aligned}$$

Example 3 : Find the lowest common multiple of 32 and 5.

The highest common factor of 32 and 5 is 1.

$$\begin{aligned}
32 &= 1 \times 32 \\
5 &= 1 \times 5 \\
\text{LCM} &= 1 \times 32 \times 5 \\
\text{LCM} &= 160
\end{aligned}$$

Example 4 : Find the lowest common multiple of 18, 24 and 36. The highest common factor of 18, 24, and 36 is 6:

$$\begin{aligned}
18 &= 6 \times 3 \\
24 &= 6 \times 2 \times 2 \\
36 &= 6 \times 2 \times 3
\end{aligned}$$

Note that one of the missing partners is 3 and is repeated in the factors of 18 and 36. We include it once only in the factors of the LCM.

$$\text{LCM} = \overbrace{6 \times 3}^{18} \times 4$$

Therefore the LCM is 72.

Exercises:

1. Find the lowest common multiple of each of the following pairs of numbers:
   (a) 24 and 30
   (b) 18 and 32
   (c) 27 and 93

Exercises 1.2 Factorization of Integers

1. (a) Is 4 a factor of 36? Explain your answer.
   (b) Is 5 a factor of 36? Explain your answer.
   (c) Write down the factors of

      i. 10 ii. 7 iii. 25 iv. 100

   (d) Find the prime factorization of

      i. 18 ii. 315

   (e) Factorize completely

      i. 126 ii. 144 iii. 100 iv. 63

   (f) Find the highest common factor of

      i. 5 & 6
      ii. 12 & 45
      iii. 120 & 63
      iv. 33 & 99

   (g) Find the lowest common multiple of

      i. 5 & 6
      ii. 12 & 45
      iii. 120 & 63
      iv. 33 & 99

2. (a) Is every number a factor of itself?
   (b) Is there a number which is a factor of all numbers?
   (c) What two numbers are factors of all even numbers?
   (d) Write down two numbers which have 2 and 5 as factors.
   (e) Suppose that 35 is a factor of a number. What two other numbers must be factors of the same number?
   (f) A number has factors including 2 and 3. What other number must also be a factor of the same number?
   (g) True or false? The lowest common multiple of 5 and 7 is the smallest number divisible by both 5 and 7?
   (h) True or false? A common multiple of two numbers is always divisible by both the numbers.
   (i) True or false? The lowest common multiple of two numbers is divisible by the highest common factor of the two numbers.
   (j) True or false? The lowest common multiple of 6 and 8 is 48.

Answers 1.2

Section 1

1. (a) 1, 2, 3, 6, 9, 18
   (b) 1, 2, 3, 4, 6, 8, 12, 24
   (c) 1, 2, 4, 7, 8, 14, 28, 56
   (d) 1, 2, 4, 8, 16, 32, 64, 128

Section 2

1. (a) $84 = 2 \times 2 \times 3 \times 7$
   (b) $72 = 2 \times 2 \times 2 \times 3 \times 3$
   (c) $126 = 2 \times 3 \times 3 \times 7$
   (d) $480 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5$
   (e) $26 = 2 \times 13$

Section 3

1. (a) 6 (b) 15 (c) 4 (d) 2 (e) 1

Section 4

1. (a) 120 (b) 32×9 (c) 31×9

Exercises 1.2

1. (a) Yes, as $4 \times 9 = 36$
   (b) No as multiples of 5 end in 0 or 5.
   (c) i. 10, 5, 2, 1
       ii. 7, 1
       iii. 25, 5, 1
       iv. 100, 50, 25, 20, 10, 5, 4, 2, 1
   (d) i. $18 = 2 \times 3 \times 3$

    ii. $315 = 3 \times 3 \times 5 \times 7$

(e) i. $126 = 2 \times 3 \times 3 \times 7$
    ii. $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$
    iii. $100 = 2 \times 2 \times 5 \times 5$
    iv. $63 = 3 \times 3 \times 7$

(f) i. 1
    ii. 3
    iii. 3
    iv. 33

(g) i. 30
    ii. 180
    iii. 2520
    iv. 99

2. (a) Yes, though not a *proper* factor
   (b) Yes, 1
   (c) 1 and 2
   (d) 10 and 20
   (e) 7,5
   (f) 6
   (g) True
   (h) True
   (i) True
   (j) False