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REVIEWS

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Notes on Fermat's Last Theorem. By Alf van der Poorten.
Wiley-Interscience, 1996, xv+222 pp., $49.95.

Reviewed by **Andrew Granville**

Have you ever wanted a math book that you could dip into like a favorite, inspired novel? One in which every page has a delicious quote, a provoking viewpoint, or a novel insight? A book which when read for the third time still makes you think or smile? That you can't put down, finding yourself reading on, even when you only picked it up to check up on one little fact? This is Van der Poorten's polished, eccentric, opinionated and inspiring *Notes on Fermat's Last Theorem.* We need more mathematics books like this.

Van der Poorten has written a book to inspire as many mathematicians as possible to enjoy the wonderful ideas that make up the background of modular forms and elliptic curves, and, in the process, much of mainstream number theory. He doesn't attempt to be complete, but instead tries to explain the flavor of much of what goes on:

> One of the difficulties in reading, or in listening to, mature mathematics is its immense vocabulary and the volume of notions that seems to be required. Nor can one readily discover the meaning of the more popular ideas because all too often they are defined in terms of yet more obscure words. The truth is, unfortunately, that few — perhaps none — of us know all the definitions. We rely on a feeling for what must be intended, knowing that we can refine that feeling should needs be. In a sense, these notes should be seen precisely as an attempt to create some useful feelings.
>
> The style adopted in the notes is to *announce* all sorts of things. Some announcements are just definitions, others are facts whose explanations we are not yet in a position to comprehend. But many of my claims are indeed obvious after one has thought a little while ...

Reading Van der Poorten is a bit like hearing a great colloquium in which you grasp the point of research in a field distant from your own, in part because the speaker astutely judges the correct amount of detail to present to persuade and interest you, and in part because the speaker assumes a level of rigour that allows you to follow and yet trust in what is going on, without being overwhelmed.

Here's an example from Lecture VII: Van der Poorten wishes to explain to the reader how we know that $\sin \pi z$ is periodic, given only that we have certain functional theoretic properties of it, evidently so that he can later develop the techniques to understand 'elliptic functions' (which are periodic in two directions on the plane). He writes

> ... we admit that $\sin \pi z$ has simple zeros exactly at $0, \pm 1, \pm 2, \ldots,$ and — rather wildly thinking of it as just a polynomial of infinite degree — we factorize it and write
>
> $$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

> Of course that multiplier π (which, after all, might have been any decent function that never vanishes) needs rather calmer justification.

In a footnote, Van der Poorten provides this calmer justification by proving that a set of points, dense on $[-1,1]$, satisfy this equation. He then proceeds:

> With this evil deed done, we acknowledge that we are frightened of products, so we take the logarithm; and being bothered by logarithms, we differentiate. That yields
>
> $$\pi \cot \pi z = \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n-z} - \frac{1}{n+z} \right).$$
>
> Unfortunately, as we catch our breath, we see that this is a mildly nasty partial fraction expansion in that it only converges conditionally — that is, on condition that we don't muck about with those parentheses. So we differentiate again and contemplate
>
> $$\pi^2 \text{cosec}^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(n-z)^2}$$
>
> and see that it shouts its periodicity. If we now backtrack carefully, we are done.

Much important technical mathematics is covered here, rigorously (though not pedantically!), but at the same time with an attempt to draw the reader's attention to the flow of the argument, rather than to distract the reader with the details. I find that this style draws me on, as a reader, inviting me to continue beyond what I know, on an easy trust, since I feel that the justification is there in a form that I can later go back to and try to understand.

To me this is in marked contrast to most mathematics books of today, which almost universally suffer from the Bourbakiist school of thought that everyone must speak the same highly technical language to appreciate what is going on (and so, those who have not yet done so, are doomed to not understand what is going on). After reading Van der Poorten's book you can't help asking yourself why we have to master so much background material, presented in as dry a manner as possible, to get at the root of what is going on in most subjects? Why do so many authors act is if mathematics has to be such a very serious business, with every "i" needing to be dotted for it to be correct? That people might learn more, and more quickly, by being excited and by being inspired, and challenged to think about the questions of the day, has escaped this strangely dominant school of educational thought. However it hasn't escaped Van der Poorten's notice. His choice is to select topics that are fun, that give the flavour to some of the meat of the subject, and yet can be explained in a series of short chapters. Sometimes he explains something in an elementary way that needs a little more justification later, or can be most readily justified in a not entirely rigorous way, but he candidly admits to these sins, and it makes me want to fill in the gaps, not to despair.

Most of the cognescenti have shied away from writing books on this topic because of the difficulty of making such technical material accessible to the non-expert, while still doing justice to the material. A difficult task indeed, and all the other such books I have read fail dismally. However Van der Poorten is perhaps the first such author to grasp even the basic material well enough to have the confidence to decide on a consistent, and plausible, perspective. What he does is to provide much of the basic background material, and the flavour of the some of the less basic, without getting gum on his shoes, by being constructive, and sometimes intentionally redundant to highlight an idea. Actually, Van der Poorten puts it well:

> I proceed to mention all sorts of odds and ends in an effort to sneak up on Wiles' argument without becoming too tangled in incomprehensible detail ... . The point is to glimpse all sorts of exciting pieces of mathematics and to be moved to teach ourselves more. Among my motives in giving these lectures was that of trying to make mathematics a little less boring. All too often the reason for the incomprehensible things one is asked to learn is "beyond the scope of the text". That seems a constipated approach to me ... . My idea was to provide motive — and damn the details.

It would be so beneficial if more authors dared to write in this way, but how can we re-direct our mathematics culture so that this would seem to be less the extreme and more the norm?

All-in-all Van der Poorten's approach reminds me of when an excited colleague, explains to me her latest research work in a field remote from mine, not being shy to discuss those little details that fascinate her (which are sometimes beside the point), while all the time making sure I don't get totally lost by throwing in lots of excellent examples.

Van der Poorten does not attempt to give a complete proof of Wiles' Theorem, nor to get involved with many of the difficult technical aspects. What he does do is give a coherent overview of the proof, digging deep enough to include some essence from the harder mathematics involved, and to introduce various fundamental questions that arise. It is enough to get you started if you intend to go on to master the details.

The first few chapters of the book discuss much of the early history of number theory, in the guise of its relationship to Fermat's Last Theorem. Thus Van der Poorten covers much of the material central to Ribenboim's classic book [1], but also gets to describe some of his favourite topics, such as continued fractions and p-adic numbers. In chapter six he starts in on the modern approaches to Fermat's Last Theorem, introducing Mordell's and Faltings' Theorems, the *abc*-conjecture, and even a first shot at explaining the Birch-Swinnerton Dyer conjectures. In chapters seven and ten, he introduces elliptic functions and Weierstrass parametrizations (including some nifty little tricks I'd not seen before), and some of the theory of Eisenstein series and modular forms. In the meantime in chapter nine he gives enough of the basics of reductions of elliptic curves that he can explain the modularity conjecture *accurately* in chapter eleven.

In chapter twelve he gives a lovely introduction to Poisson summation, which allows him to deduce functional equations and explain some of why they are interesting. In chapter thirteen he gives a more detailed discussion of L-functions and their role in modern mathematics. In chapter fifteen Van der Poorten discusses heights. This is a beautiful section, giving clear motivation to view several notions of height as aspects of the same idea, proceeding from Mahler measure to canonical height. This then allows him to give a more complete explanation of the Birch-Swinnerton Dyer conjectures in chapter sixteen. By describing the construction of Heegner points, Van der Poorten then explains the Gross-Zagier formula, and so motivates the solution to Gauss's class number problem. In chapter seventeen, Van der Poorten has a stab at explaining the relevance of Galois representations, the Deligne-Serre theorem, and thence on to his sketch of the proof of Wiles. Once you have gotten this far you are ready to move on to learning more of the details, and Van der Poorten has succeeded in his goal of getting you *involved* in the mathematics.

One beautiful aspect to this book is that Van der Poorten manages to discuss several of the more interesting recent developments in number theory, which are not necessarily closely related to Fermat's Last Theorem and the modularity conjecture. By developing number theory as he does, we see how these different questions arise naturally in their own context, and how these same kinds of tools lead into approaches to these various different problems.

Van der Poorten's book will be a special addition to your bookshelves not only for the discussion of mathematics, and for the refreshingly honest approach to how mathematics is really done, but also because he knows and discusses the people involved, he has a varied sense of humour and a freeish style of writing, and he is very aware of the cultural effect of the resolution of Fermat's Last Theorem.

Indeed, now that Fermat's Last Theorem has been proved (by the way, "Last" is as in "Last to be proved"), many people, including the *Monthly* editor Underwood Dudley who commissioned this article, are asking "What next?" What question is going to take the place of FLT, to inspire and provoke the next hundred generations of students, just as FLT did? To inspire and provoke them to explore mathematics for themselves, to experiment, to play, and to discover that the more one probes in mathematics, the more one learns that there is so much to yet be understood? There are several candidates for such a question: Old favorites like the *Riemann Hypothesis*, the *Twin Prime Conjecture*, or the *Poincaré Conjecture*, more modern questions like **P≠NP**, or the *abc-conjecture*, or off-shoots of Wiles' Theorem, like *Prove that there are no coprime positive integers* x*,* y*,* z *satisfying* $x^p + y^q = z^r$ *with* p*,* q*,* $r \geq 3$. The experts, by definition, are unlikely to predict which question will turn out to be so inspiring to the uninitiated, and how it will provoke the next Wiles into becoming a mathematical researcher. In fact I doubt that any such question will emerge in the foreseeable future.

Having been repeatedly asked this question in the last few years, I have tried to explore the impact of Wiles' extraordinary work on the culture that defines our subject. I have come to the conclusion that the proof of Fermat's Last Theorem is as much Mathematics' greatest loss as one of Mathematics' greatest wins. Fermat's Last Theorem had everything, a romantic story (the lost, and marginal, proof), easily understood background information (Pythagoras' Theorem, and 3-4-5 triangles), it was easy to take part and think of your own way to go about it (all those flawed proofs), it had a deep and rich mathematical history (Cauchy, Kummer, Vandiver, ... , Faltings, Frey, Serre, Ribet, ...), early work from one of the greatest of the early female mathematicians (Germain), and even high financial stakes (like the Wolfskehl prize)!

Now that the Holy Grail of Mathematics has been found, how else to rally the faithful to Camelot? Can the deeper, more complex and arguably more important, questions entice? Surely the uninitiated always have and always will want a glittering prize ahead of them, just out of reach, but close enough to draw them on? Perhaps it would have been better if FLT had never been resolved, if it had remained as testament to our limitations, just beyond the reach of mortal ken.

It has long been trendy to downplay FLT's importance, and to instead focus on deeper, less immediately enticing, questions. Indeed Kummer called FLT "more of a joke than a pinnacle of science", while Gauss would not deign to work on it (or so he claimed!). However I believe that behind that sophisticated façade, both would have been delighted to resolve the question (clearly much of Kummer's greatest work was motivated by his study of FLT). Several of today's expert naysayers, who were scathing about the importance of FLT not so long ago, put aside their prejudices when the time came, and enthusiastically rejoiced in the amazing conclusion to the Fermat story. This was fitting, for Wiles' great proof is a true milestone in the history of mathematics: It exhibits how so many of the abstract developments of mathematics influence the simplest of all serious questions, and perhaps let us believe that so much of the work that has been done, in so many diverse areas, is really worthwhile!

How can any other question reflect so well the mathematical culture from which it springs, and in which it is finally laid to rest?

Finally, let me repeat that the monograph under review is a wonderful mathematics book, daring to breach the stylistic barriers that usually impede understanding. It is written to encompass a lot of material, from most elementary to very deep, while remaining accessible. I expect it will turn a lot of people on to number theory and arithmetic geometry, and indeed the beauty of mathematics as a whole. At the very least, if you have a clever undergraduate student, bored by upper division calculus and ready for something a little more poignant, get her to read this book and let her first experience of research level mathematics be provoking, inspiring and fun.

REFERENCES

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