Periodic Continued Fractions in Elliptic Function Fields

Alfred J. van der Poorten and Xuan Chuong Tran

ceNTRe for Number Theory Research,
Macquarie University, Sydney,
NSW 2109, Australia
alf@mpce.mq.edu.au (Alf van der Poorten)
xctran@hotmail.com (Xuan Chuong Tran)

Abstract. We construct all families of quartic polynomials over $\mathbb{Q}$ whose square root has a periodic continued fraction expansion, and detail those expansions. In particular we prove that, contrary to expectation, the cases of period length nine and eleven do not occur. We conclude by providing a list of examples of pseudo-elliptic integrals involving square roots of polynomials of degree four. The primary issue is of course the existence of units in elliptic function fields over $\mathbb{Q}$. That, and related issues are surveyed in the paper's introduction.

1 Introduction

We provide the expansion of all families of quartic polynomials defined over $\mathbb{Q}$ and with periodic continued fraction expansion, and derive from that a list of examples of each family of pseudo-elliptic integrals involving square roots of such polynomials of degree four.

2 Units in Quadratic Function Fields

Let $D(X)$ be a polynomial, not a square, defined over a field $\mathbb{F}$ of characteristic zero, and suppose there are polynomials $x(X)$, $y(X)$ defined over $\mathbb{F}$, with $y \neq 0$, so that $x^2 - Dy^2$ is a constant $-\kappa$, of course in $\mathbb{F}$.

Example 1. Suppose we are given the pseudo-elliptic integral

$$\int^u \frac{4t-1}{\sqrt{t^4-2t^3+3t^2+2t+1}}\,dt$$
$$= \log\bigl((u^4-3u^3+5u^2-2u)+(u^2-2u+2)\sqrt{u^4-2u^3+3u^2+2u+1}\,\bigr).$$

Set $D(u) = u^4-2u^3+3u^2+2u+1$, $x(u) = u^4-3u^3+5u^2-2u$, $y(u) = u^2-2u+2$. We may save ourselves an annoying verification. Add to the given claim the corresponding allegation with $\sqrt{D}$ replaced by $-\sqrt{D}$. On the left we integrate zero, and on the right we obtain $\log(x^2-Dy^2)$; that is, $x^2-Dy^2=-\kappa$ must be a constant.

Example 2. Because $D \neq \square$, it is plain that $\kappa \neq 0$. Just so, D must be of even degree, $2g + 2$ say, and with leading coefficient a square in $\mathbb{F}$. It follows that $\delta(X) = \sqrt{D(X)}$ is represented by a Laurent series in $\mathbb{F}((X^{-1}))$, say $\sum_{h=-g-1}^{\infty} d_h X^{-h}$.

Take $\delta(X) = \sqrt{X^4 - 2X^3 + 3X^2 + 2X + 1}$. Then

$$\begin{aligned}\delta(X) = X^2 - X + 1 + 2X^{-1} + 2X^{-2} - 4X^{-4} - 8X^{-5} - 6X^{-6} + 10X^{-7} \\ + 40X^{-8} + 58X^{-9} + 2X^{-10} - 188X^{-11} - 442X^{-12} - 382X^{-13} + \cdots \quad .\end{aligned}$$

Plainly, the element $u = x - \delta y$ of the function field $\mathbb{K} = \mathbb{F}(X, \delta)$ of the curve $\mathcal{C} : Y^2 = D(X)$ is a non-trivial unit in $\mathbb{K}$. Indeed, it divides a trivial unit $\kappa \in \mathbb{F} \subset \mathbb{K}$. Hence the divisor of u on the Jacobian $\operatorname{Jac}(\mathcal{C})$ of $\mathcal{C}$ is supported only at infinity, thus at just two points, which we may conveniently call ∞_+ and ∞_-. Because it is the divisor of a function it has degree zero and thus there is some integer m — in fact, the regulator of $\mathbb{K}$ — so that $m(\infty_+ - \infty_-)$ is the divisor of a function. That is, $\infty_+ - \infty_-$ is a torsion point of order m on $\operatorname{Jac}(\mathcal{C})$.

It is well understood that the existence of a non-trivial unit in $\mathbb{K}$ guarantees that δ has a periodic continued fraction expansion. In [11] we also explain why, unlike the case of real quadratic irrationals where the continued fraction of the square root $\sqrt{D}$ of any positive nonsquare integer is always periodic, the continued fraction of the square root $\delta(X)$ of a polynomial D is not always periodic. The point is that, by the box principle, Pell's equation $x^2 - Dy^2 = 1$ always has a solution in the number case, but — because there are infinitely many polynomials of bounded degree if the base field $\mathbb{F}$ is infinite — Pell's equation does not necessarily have a solution in the function case. Assisted by ideas of Tom Berry [2], we also detail the structure of the period of the continued fraction expansion of $\sqrt{D(x)}$ when D is a polynomial over a field $\mathbb{F}$ and the expansion of $\sqrt{D(x)}$ happens to be periodic. In particular, we notice that, given the existence of unit $x - \delta y$ with norm $x^2 - Dy^2 = -\kappa$, then $\big((x^2 + Dy^2) - 2\delta xy\big)/\kappa$ is a unit of norm 1, given by a period of the continued fraction expansion of δ. For $\kappa \neq -1$, the unit $x - \delta y$ is said to be given by a *quasi*-period.

We recall that a continued fraction expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \ddots}}}}}$$

plainly needs a less wasteful notation, say $[\,a_0\,,\,a_1\,,\,a_2\,,\,\ldots]$, to represent it.

Example 3. We have

$$\begin{aligned}&\sqrt{X^4 - 2X^3 + 3X^2 + 2X + 1} = \\ &[X^2 - X + 1, \\ &\quad \overline{\tfrac{1}{2}X - \tfrac{1}{2},\ 2X - 2,\ \tfrac{1}{2}X^2 - \tfrac{1}{2}X + \tfrac{1}{2},\ 2X - 2,\ \tfrac{1}{2}X - \tfrac{1}{2},\ 2X^2 - 2X + 2}],\end{aligned}$$

displaying the full period, whereas

$$[X^2 - X + 1\,,\ \tfrac{1}{2}X - \tfrac{1}{2}\,,\ 2X - 2] = x(X)/y(X),$$

with $x(X) = X^4 - 3X^3 + 5X^2 - 2X$ and $y(X) = X^2 - 2X + 2$, already provides a unit, $x(X) - y(X)\sqrt{X^4 - 2X^3 + 3X^2 + 2X + 1}$, of norm -4. One also notices

$$4 \cdot [\tfrac{1}{2}X - \tfrac{1}{2}\,,\ 2X - 2\,,\ \tfrac{1}{2}X^2 - \tfrac{1}{2}X + \tfrac{1}{2}\,] = [2X - 2\,,\ \tfrac{1}{2}X - \tfrac{1}{2}\,,\ 2X^2 - 2X + 2\,].$$

Here, we recall Wolfgang Schmidt's felicitous formulation [15] of a well known fact:

Proposition 1 (Multiplication of continued fractions by a constant).

$$B[\,Ca_0\,,\, Ba_1\,,\, Ca_2\,,\, Ba_3\,,\, Ca_4\,,\, \ldots] = C[\,Ba_0\,,\, Ca_1\,,\, Ba_2\,,\, Ca_3\,,\, Ba_4\,,\, \ldots].$$

Example 4. It is a consequence of the various symmetries and twisted symmetries possessed by the period of the the square root of a quadratic irrational with polynomial trace (such symmetries are instanced by the preceding example), that a quasi-period must be of odd length; that is, if it is of even length then it is in fact a period.

Look carefully at the period $a_1\,,\, a_2\,,\, \ldots\,,\, a_{2r}$:

$$\tfrac{1}{2}X - \tfrac{1}{2}\,,\ 2X - 2\,,\ \tfrac{1}{2}X^2 - \tfrac{1}{2}X + \tfrac{1}{2}\,,\ 2X - 2\,,\ \tfrac{1}{2}X - \tfrac{1}{2}\,,\ 2X^2 - 2X + 2\,.$$

Other than for $2r = 6$ and $\kappa = 4$, the following features are not particular to the example. First, the word $a_1 a_2 \cdots a_{2r-1}$ is symmetric. Second, as 'also noticed' above, the second half repeats the first half of the period, up to a twist by κ. In the example, r is too small fully to illustrate that the half period $a_1 \cdots a_{r-1}$ is twisted symmetric: in that $\kappa \cdot [\,a_1\,,\, \ldots\,,\, a_{(r-1)/2}\,] = [\,a_{r-1}\,,\, \ldots\,,\, a_{(r+1)/2}\,]$. Whatever, these observations force r indeed to be odd.

The non-periodic case is considered in [12]. There, the point is that it is easy enough to notice periodicity, but not at all obvious how to prove non-periodicity. Aided by remarks of Jin Yu in [17], the paper [12] instances a simple criterion (based on reduction modulo different primes) that readily allows the detection of non-periodicity from inspection of just several initial partial quotients of the continued fraction expansion.

Below, we apply the results alluded to above to compute all quartic polynomials $D(x)$ over $\mathbb{Q}$ so that $\sqrt{D(x)}$ does have a periodic continued fraction expansion. In the case $\deg D = 4$, the curve $\mathcal{C} : Y^2 = D(X)$ is of genus $g = 1$, and may be considered to coincide with its Jacobian. Thus it suffices to list the various possibilities for the order of torsion points on an elliptic curve, as we may by a celebrated result of Mazur [7], and, following the algorithm given by Adams and Razar [1], to obtain the model $\mathcal{C}$ so as to have located the relevant torsion point at infinity. Specifically, given an elliptic curve $E/\mathbb{Q} : v^2 = u^3 + Au + B$ and a rational point $P = P(a, b)$ on E, the transformation

$$u = \tfrac{1}{2}(X^2 + Y - a), \qquad v = \tfrac{1}{2}(X^3 + XY - 3aX - 2b) \tag{1}$$

maps P and the point at infinity O on E to the two points at infinity on

$$E_P : Y^2 = X^4 - 6aX^2 - 8bX + c\,,$$

where $c = -4A - 3a^2$ and $B = b^2 - a^3 - Aa$.

Conversely the formulas

$$X = (v+b)/(u-a)\,, \qquad Y = 2u + a - \big((v+b)/(u-a)\big)^2$$

transform the quartic model E_P back to E; thus (1) is a birational transformation.

The elliptic case $g = 1$ is congenial for reasons additional to Mazur's theorem. For general genus g, it is easy to see that the complete quotients $\delta_h(X)$ of δ are all of the shape

$$\delta_h = (P_h + \sqrt{D})/Q_h\,,$$

with $Q_h \big| D - P_h^2$ and, this remark is in part just setting the notation, the generic step in the continued fraction algorithm for $\delta = \sqrt{D}$ is

$$\delta_h = (P_h + \sqrt{D})/Q_h = a_h - (P_{h+1} - \delta)/Q_h\,. \tag{2}$$

Here the sequences of polynomials (P_h) and (Q_h) are given sequentially by

$$P_{h+1} + P_h = a_h Q_h, \quad \text{and} \quad Q_{h+1}Q_h = D - P_{h+1}^2\,.$$

Proposition 2. *The polynomials Q and P satisfy* $\deg Q_h \le g = \frac{1}{2}\deg D - 1$ *and* $\deg P_{h+1} = g + 1 = \frac{1}{2}\deg D$ *for all* $h = 0, 1, \ldots$.

Proof. Given $\deg Q_h \le g$ it follows from $-(P_{h+1} - \sqrt{D})/Q_h$ being a remainder, so that it is of negative degree, that $\deg P_{h+1} = g + 1$ and $\deg(P_{h+1} - \sqrt{D})$ is less than $\deg Q_h$. Thus $Q_{h+1}Q_h = D - P_{h+1}^2$ entails that $\deg Q_{h+1} \le g$. Finally, $\delta_0 = \sqrt{D}$ displays that $Q_0 = 1$, so $\deg Q_0$ is no more than g.

Now notice that $P_{h+1} + P_h = a_h Q_h$ entails that $\deg Q_h = 0$ is equivalent to $\deg a_h = g+1$. However, $\deg Q_l = 0$ signals that δ has a quasi-period comprising the partial quotients $a_1, a_2, \ldots, a_l$. Moreover, if this is a primitive such period then, other than for a_l, all those partial quotients have degree at most g. Thus, in the elliptic case, the quasi-period length l implies that the regulator m — the degree of the fundamental unit or, equivalently, the sum of the degrees of the partial quotients comprising the quasi-period — is given by $m = l+1$. For larger g, the corresponding argument typically does no better than $m \ge l + g$.

Back to the case $\deg D = 4$ and base field $\mathbb{Q}$, we know from [7] that the possible values for m are 2, 3, …, 10, and 12; because those are the possible torsion orders of the 'divisor at infinity' on $\mathcal{C}$.

We recall that a quasi-period of even length is in fact a period, whereas a quasi-period of odd length r might be a period, or it yields a primitive period of length $2r$. It follows from the first reason that we will find primitive periods of length 2, 4, 6, and 8, and for the second reason that there surely will be primitive periods of length 1 and 2, 3 and 6, 5 and 10, 7 and 14, 9 and 18, and 11 and 22. Here one expects the periods of odd length to occur because the norm of the fundamental unit may surely happen to be -1.

However, as it happens, we see below that the periods 9 and 11 do *not* occur.

Example 5. Set $D(X) = X^4 - 2X^3 + 3X^2 + 2X + 1$, and consider the continued fraction expansions of the numbers $\sqrt{D(n)}$ for $n = 1, 2, \ldots$. Of course these expansions are periodic, of respective period lengths $\ell\big(D(n)\big) = \ell_n$, say. It is notorious that, given an arbitrary positive integer k, not a square, it is in general extraordinarily difficult to predict the period length $\ell(k)$ of the expansion of $\sqrt{k}$. Yet here $\ell_{2n-1} = 17$ and $\ell_{2n} = 7$ for $n = 2, 3, \ldots$. By the way, all the $D(n)$ are 1 modulo 4 so that, in decency, we should have considered the quantities $(\sqrt{D(n)} + 1)/2$ in place of $\sqrt{D(n)}$. Indeed, their periods all have length 5, for $n = 2, 3, \ldots$.

This last remark is apropos, given a theorem of Schinzel [16] to the following effect. Suppose $f(X)$ is a polynomial, not a square, taking positive integer values at $X = 1, 2, \ldots$. Denote by ℓ_n the length of the period of the continued fraction expansion of $\sqrt{f(n)}$. Then $\limsup_{n\to\infty} \ell_n$ is finite if and only if there is a nontrivial unit in the function field $\mathbb{Q}\big(X, \sqrt{f(X)}\,\big)$, which moreover has *integer* coefficients, that is there is a unit defined over $\mathbb{Z}\big(X, \sqrt{f(X)}\,\big)$.

In this context, Schinzel speculates on the possible period lengths for quartic polynomials f ([16, p297]) reporting 1 and the even lengths “and possibly also 5, 7, 9, 11 (I have not verified this) … ”. Of course, in 1962 the result of Mazur was as yet no more than a conjecture (of Nagell). Related remarks of Schinzel include essentially everything observed above and make clear moreover that these things were mostly already known to Abel and Tchebicheff. For details and references see [16, II §4]. The continued fractions in the easier genus zero case are given by [16, I] and are discussed by van der Poorten and Hugh Williams in [13].

Pseudo-elliptic integrals, as instanced at Example 1, are the subject of [11]; with one change. In [11] we write about *quasi*-elliptic integrals as if these integrals are ‘sort of’ elliptic, in the sense that a quasi-period certainly kind of is a period (quasi: resembling; as it were …). The qualifier *quasi* was incorrect. It would have been more to the point to speak of *pseudo*-elliptic integrals (pseudo: a word element meaning false, pretended …), emphasising that these integrals have elliptic appearance but are not elliptic at all.

3 Continued Fractions of Quadratic Irrationals

Anyone attempting to compute the truncations $[\,a_0\,, a_1\,, \ldots\,, a_h\,] = x_h/y_h$ of a continued fraction will be delighted to notice that the definition

$$[\,a_0\,, a_1\,, \ldots\,, a_h\,] = a_0 + 1/[\,a_1\,, \ldots\,, a_h\,]$$

immediately implies by induction on h that there is a correspondence

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_h & x_{h-1} \\ y_h & y_{h-1} \end{pmatrix} \longleftrightarrow [\,a_0\,, a_1\,, \ldots\,, a_h\,] = x_h/y_h$$

between products of certain two by two matrices and the convergents of continued fractions. Notice, incidentally, that if a product of matrices corresponds to x_h/y_h then so does any nonzero polynomial multiple of that product of matrices.

Proposition 3 below is discussed in [11].

Proposition 3. *Let δ be a quadratic irrational function with trace t and norm n both polynomials; that is, $\delta^2 - t\delta + n = 0$. Suppose x and y are polynomials so that the matrix*

$$M = \begin{pmatrix} x & -ny \\ y & x - ty \end{pmatrix}$$

has determinant $(x - \delta y)(x - \overline{\delta} y) = (-1)^r \kappa$, with κ a nonzero constant. Then M has a unique decomposition

$$M = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{r-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (a-t)/\kappa & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix},$$

where a, a_1, ..., a_{r-1} are polynomials of degree at least one satisfying $a_1 = \kappa a_{r-1}$, $a_2 = a_{r-2}/\kappa$, $a_3 = \kappa a_{r-3}$, ... Hence, if r is even then $\kappa = 1$. Moreover

$$\delta = [\, a \,,\, \overline{a_1 \,,\, \ldots \,,\, a_{r-1} \,,\, (2a-t)/\kappa \,,\, \kappa a_1 \,,\, \ldots \,,\, a_{r-1}/\kappa \,,\, 2a - t} \,] \tag{3}$$

provides the periodic continued fraction expansion of δ.

Of course, if $\kappa = 1$ then δ has period length r rather than $2r$.

Proposition 4. *If δ has quasi-period length r, but period length $2r$ — thus $\kappa \neq 1$ and r is odd — then $\mu\delta$ has period length r if and only if $\mu^2 = 1/\kappa$.*

Proof. Take δ as in (3). By Proposition 1 we see that

$$\mu\delta = [\, \mu a \,,\, \overline{a_1/\mu \,,\, \ldots \,,\, \mu a_{r-1} \,,\, (2a-t)/\mu\kappa \,,\, \kappa a_1/\mu \,,\, \ldots \,,\, a_{r-1}/\mu\kappa \,,\, \mu(2a-t)} \,],$$

so indeed $\mu = 1/\mu\kappa$ is of the essence.

4 Elliptic Curves with Torsion at Infinity

We recall Mazur's theorem limiting the possible rational torsion on a elliptic curve defined over $\mathbb{Q}$.

Proposition 5 (Mazur). *If E is an elliptic curve defined over $\mathbb{Q}$, then the torsion subgroup $E(\mathbb{Q})_{tors}$ of $E(\mathbb{Q})$ is isomorphic to either*

$$\begin{array}{lll} & \mathbb{Z}_m & \textit{for} \quad m = 1,\, 2,\, 3,\, \ldots,\, 10,\, 12 \\ \textit{or} & \mathbb{Z}_2 \times \mathbb{Z}_m & \textit{for} \quad m = 2,\, 4,\, 6,\, 8. \end{array}$$

Thus for each $m \in \{2, 3, \ldots, 10, 12\}$ we need all curves $\mathcal{C}_m : Y^2 = D_m(X)$ with D_m a polynomial of degree 4 and defined over $\mathbb{Q}$ and so that $\mathcal{C}_m$ has a torsion point of order m at infinity, equivalently — see page 103 — so that the continued fraction expansion of $\sqrt{D}$ is periodic with quasi-period length $m - 1$. Naturally we lose no generality in normalising so that D_m is monic and has zero trace.

4.1 Tabulations

The first tabulation of rational elliptic curves with given torsion group[1] probably is given by Kubert [6]. Table 3 of [5], copied below, provides a congenial version of Kubert's table, listing in Tate normal form all elliptic curves

$$E : y^2 + (1 - c)xy - by = x^3 - bx^2 \tag{4}$$

with $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}_m$ ($m = 4, 5, \ldots, 10, 12$) and $E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2m}$ ($m = 2, 3, 4$); in each case the point $(0, 0)$ is a torsion point of maximal order.

$E(\mathbb{Q})_{\text{tors}}$	b	c
$\mathbb{Z}_4$	t	0
$\mathbb{Z}_5$	t	t
$\mathbb{Z}_6$	$t(t+1)$	t
$\mathbb{Z}_7$	$t^2(t-1)$	$t(t-1)$
$\mathbb{Z}_8$	$(t-1)(2t-1)$	$\dfrac{(t-1)(2t-1)}{t}$
$\mathbb{Z}_9$	$t^2(t-1)(t^2-t+1)$	$t^2(t-1)$
$\mathbb{Z}_{10}$	$\dfrac{t^3(t-1)(2t-1)}{(t^2-3t+1)^2}$	$-\dfrac{t(t-1)(2t-1)}{t^2-3t+1}$
$\mathbb{Z}_{12}$	$\dfrac{t(2t-1)(2t^2-2t+1)(3t^2-3t+1)}{(t-1)^4}$	$-\dfrac{t(2t-1)(3t^2-3t+1)}{(t-1)^3}$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\frac{1}{16}(4t-1)(4t+1)$	0
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$-\dfrac{2(t-1)^2(t-5)}{(t^2-9)^2}$	$-\dfrac{2(t-5)}{t^2-9}$
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\dfrac{(2t+1)(8t^2+4t+1)}{(8t^2-1)^2}$	$\dfrac{(2t+1)(8t^2+4t+1)}{2t(4t+1)(8t^2-1)^2}$

One notices that the cases $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$ are just special cases of torsion order $2m$; thus, in the sequel, we will not need the last three lines of the table.

With the change of variables $x = u^2x' + r$, $y = u^3y' + su^2x' + t$, where

$$\begin{cases} u = 1, & r = -(c^2 - 2c - 4b + 1)/12, \\ s = (c-1)/2, & t = -(c^3 - 3c^2 - (4b-3)c - (8b+1))/24, \end{cases}$$

we see that the elliptic curve (4) is isomorphic to

$$E: \ y^2 = x^3 - (c_4/48)x - c_6/864\,,$$

[1] At the time, the fact that [6] provided a complete list was of course only conjectural.

where c_4 and c_6 is standard notation for the invariants of the curve (4); see for example [4], or [3]. The point $(0,0)$ is transformed to

$$P = \big((c^2 - 2c - 4b + 1)/12,\ -b/2\big)$$

and, by isomorphism, is a torsion point P on E with maximal order.

The reader interested in constructing a table of rational torsion types such as the one above will find valuable instruction in the papers [8] and [9] of Nitaj.

4.2 Quartic Coverings of Elliptic Surfaces

Finally, the transformation (1) recommended by Adams and Razar [1], see page 102, provides a list of quartic covers

$$\mathcal{C}_m(s) : Y^2 = D_m(X,s) = \big(X^2 + u_m(s)\big)^2 + v_m(s)\big(X + w_m(s)\big) \qquad (5)$$

defined over $\mathbb{Q}(s)$ so that the divisor at infinity on the Jacobian of the curve $\mathcal{C}_m$ is torsion of order m (in brief, so that the point at infinity on the curve is torsion of order m). Here s (which replaces the t of the table for elegant variation) is a parameter ranging over $\mathbb{Q}$ omitting only several isolated values. One checks readily that the continued fraction expansion of Y begins $[\, X^2 + u\ ,\ 2(X - w)/v\ ,\ \dots\,]$.

We use just brute force to notice that if $m = 2$ then the continued fraction expansion is

$$[\, X^2 + s\ ,\ \overline{2(X^2 + s)/t\ ,\ 2(X^2 + s)}\,]$$

and necessarily

$$\mathcal{C}_2(s,t) : Y^2 = D(X,s,t) = \big(X^2 + s\big)^2 + t, \qquad s \in \mathbb{Q},\ t \in \mathbb{Q} \setminus \{0\}. \qquad (6)$$

The special case $t = 1$ gives period length $r = 1$.

Similary, if $m = 3$ then the continued fraction expansion must be

$$[\, X^2 - s^2\ ,\ \overline{2(X + s)/t\ ,\ 2(X^2 - s^2)}\,]$$

and so

$$\mathcal{C}_3(s,t) : Y^2 = D(X,s,t) = \big(X^2 - s^2\big)^2 + t(X - s), \qquad s \in \mathbb{Q},\ t \in \mathbb{Q} \setminus \{0\}. \quad (7)$$

In all other cases we obtain an elliptic surface $D_m(X,s)$ thus with just one rational parameter.

Here and below, we detail only the continued fraction expansions, seemingly breaking the cardinal rule that when dealing with quadratic irrationals one must mind one's P's and Q's. That is, the critical information is contained in the complete quotients $(Y_m + P_h)/Q_h$, rather than in the partial quotients a_h. However, here we lose no information to speak of. The reader can readily confirm that a partial quotient $2(X - c_h)/b_h$ entails that $Q_h = b_h(X + c_h)$, and if $P_h = X^2 + u_m + 2e_h$ then $e_{h+1} = -(e_h + u_m + c_h^2)$. Of course, the partial quotient $2(X^2 + u_m)/k_m$ implies $Q = k_m$ and $e = u_m$. We take $P_0 = 0$ and $Q_0 = 1$ but, in decency, we ought to be expanding $Y_m + (X^2 + u_m)$, thus with $P_0 = X^2 + u_m$. Note that, in any case, $P_1 = X^2 + u_m$, that is, $e_1 = 0$.

4.3 Periods of Even Length

We summarised the case $m = 3$ at (7) on page 107. The case $m = 5$ is

$$\begin{aligned}\mathcal{C}_5(t) : Y_5^2(X,t) &= D(X,t)\\ &= \bigl(X^2 - \tfrac14(t^2-6t+1)\bigr)^2 + 4t\bigl(X - \tfrac12(t-1)\bigr)\,, \quad t \in \mathbb{Q}\setminus\{0\}, \qquad (8)\end{aligned}$$

with continued fraction expansion

$$Y_5(s) = [\, X^2 - \tfrac14(t^2-6t+1)\,,\ \overline{\bigl(X + \tfrac12(t-1)\bigr)/2t\,,\ 2\bigl(X - \tfrac12(t+1)\bigr)\,,\ \bigl(X + \tfrac12(t-1)\bigr)/2t\,,\ 2\bigl(X^2 - \tfrac14(t^2-6t+1)\bigr)}\,].$$

Just so, $\mathcal{C}_7(t)$ is defined by

$$u_7(t) = -\tfrac14(t^4-6t^3+3t^2+2t+1), \quad v_7(t) = 4t^2(t-1), \quad w_7(t) = -\tfrac12(t^2-t-1),$$

and $Y_7(X,t)$ has continued fraction expansion,

$$\begin{aligned}[\, X^2 + u_7(t)\,,\ &\overline{\tfrac12\bigl(X + \tfrac12(t^2-t-1)\bigr)/t^2(t-1)\,,\ 2\bigl(X - \tfrac12(t^2-t+1)\bigr)\,,}\\ &\overline{\tfrac12\bigl(X + \tfrac12(t^2-3t+1)\bigr)/t(t-1)\,,}\\ &\overline{2\bigl(X - \tfrac12(t^2-t+1)\bigr)\,,\ \tfrac12\bigl(X + \tfrac12(t^2-t-1)\bigr)/t^2(t-1)\,,\ 2\bigl(X^2+u_7(t)\bigr)}\,]. \qquad (9)\end{aligned}$$

Finally, for this is the last case with m odd, for $m = 9$ we have

$$\begin{aligned}u_9(t) &= -\tfrac14(t^6-6t^5+9t^4-10t^3+6t^2+1),\\ v_9(t) &= 4t^2(t-1)(t^2-t+1), \qquad w_9(t) = -\tfrac12(t^3-t^2-1), \qquad (10)\end{aligned}$$

with continued fraction expansion

$$\begin{aligned}[\, X^2 + u_9(t)\,,\ &\overline{\tfrac12\bigl(X + \tfrac12(t^3-t^2-1)\bigr)/t^2(t-1)(t^2-t-1)\,,}\\ &\overline{2\bigl(X - \tfrac12(t^3-t^2+1)\bigr)\,,\ \tfrac12\bigl(X - \tfrac12(t^3-3t^2+2t-1)\bigr)/t^2(t-1)\,,}\\ &\overline{2t\bigl(X - \tfrac12(t^3-3t^2+4t-1)\bigr)/(t^2-t+1)\,,}\\ &\overline{\tfrac12\bigl(X - \tfrac12(t^3-3t^2+2t-1)\bigr)/t^2(t-1)\,,\ 2\bigl(X - \tfrac12(t^3-t^2+1)\bigr)\,,}\\ &\overline{\tfrac12\bigl(X + \tfrac12(t^3-t^2-1)\bigr)/t^2(t-1)(t^2-t-1)\,,\ 2\bigl(X^2+u_9(t)\bigr)}\,].\end{aligned}$$

4.4 Periods of Odd Length

We have dealt with the case $m = 2$ at page 107. When $m = 4$ we find that

$$\mathcal{C}_4(t) : Y_4(X,t)^2 = D(X) = \bigl(X^2 + \tfrac14(4t-1)\bigr)^2 + 4t(X + \tfrac12)\,, \quad t \in \mathbb{Q}\setminus\{0\}, \qquad (11)$$

and

$$Y_4(X,t) = [\,X^2+\tfrac14(4t-1)\,,\,\overline{2(X-\tfrac12)/4t\,,\,2(X-\tfrac12)\,,\,2\big(X^2+\tfrac14(4t-1)\big)/4t\,,}$$
$$\overline{2(X-\tfrac12)\,,\,2(X-\tfrac12)/4t\,,\,2\big(X^2+\tfrac14(4t-1)\big)}\,].$$

Thus $\kappa_4(t)=4t$. This entails that $Y_4(X,\frac14 s^2)/s$ has the periodic continued fraction expansion of period length $r=3$:

$$[\,\big(X^2+\tfrac14(s^2-1)\big)/s\,,\,\overline{2(X-\tfrac12)/s\,,\,2s(X-\tfrac12)\,,\,2\big(X^2+\tfrac14(s^2-1)\big)/s}\,].$$

For $m=6$, and $t\in\mathbb{Q}\setminus\{0,-1\}$, the surface $\mathcal{C}_6(t)$ is given by

$$u_6(t)=\tfrac14(3t^2+6t-1),\quad v_6(t)=4t(t+1),\qquad w_6(t)=-\tfrac12(t-1). \tag{12}$$

and its continued fraction is detailed by

$$[\,X^2+\tfrac14(3t^2+6t-1)\,,\,\big(X+\tfrac12(t-1)\big)/2t(t+1)\,,\,2\big(X-\tfrac12(t+1)\big)\,,$$
$$\big(X-\tfrac12(t+1)\big)/2t\,,\,2\big(X+\tfrac12(t-1)\big)/(t+1)\,,$$
$$2\big(X^2+\tfrac14(3t^2+6t-1)\big)/4t\,,\,\ldots\,].$$

Thus $\kappa_6(t)=4t$. It follows that $Y_6(X,s^2)/2s$ has the periodic continued fraction expansion of period length $r=5$:

$$[\,\big(X^2+\tfrac14(3s^4+6s^2-1)\big)/2s\,,\,\overline{\big(X+\tfrac12(s^2-1)\big)/s(s^2+1)\,,}$$
$$\overline{\big(X-\tfrac12(s^2+1)\big)/s\,,\,\big(X-\tfrac12(s^2+1)\big)/s\,,}$$
$$\overline{\big(X+\tfrac12(s^2-1)\big)/s(s^2+1)\,,\,2\big(X^2+\tfrac14(3t^2+6t-1)\big)/2s}\,].$$

Finally, because this provides the last of the cases with odd period length, the elliptic surface $\mathcal{C}_8(t): Y_8^2(X,t)=D_8(X,t)$ is defined by

$$u_8(t)=(4t^4+4t^3-16t^2+8t-1)/4t^2,$$
$$v_8(t)=4(t-1)(2t-1),\qquad w_8(t)=-(2t^2-4t+1)/2t, \tag{13}$$

and, if $t\in\mathbb{Q}\setminus\{0,\frac12,1\}$, then $Y_8(X,t)$ has the continued fraction expansion

$$[\,X^2+u_8(t)\,,\,\tfrac12\big(X+(2t^2-4t+1)/2t\big)/(t-1)(2t-1)\,,$$
$$2\big(X-(2t^2-4t+1)/2t\big)\,,\,\tfrac12 t\big(X-(2t-1)/2t\big)/(t-1)(2t-1)\,,$$
$$2(2t-1)\big(X-(2t-1)/2t\big)/t^2\,,\,\tfrac12 t^3\big(X-(2t^2-4t+1)/2t\big)/(t-1)(2t-1)^2\,,$$
$$2(2t-1)\big(X+(2t^2-4t+1)/2t\big)/t^3\,,\,\tfrac12 t^3\big(X^2+u_8(t)\big)/(t-1)(2t-1)^2\,,\ldots].$$

Thus $\kappa_8(t)=4(t-1)(2t-1)^2/t^3$. It follows that $Y_8\big(X,1/(1-s^2)\big)/2s(1+s^2)$ has a continued fraction expansion with period $r=7$ for $s\in\mathbb{Q}\setminus\{0,\pm1\}$. For example

$$\tfrac{1}{20}Y_8(\tfrac16 X,-\tfrac13)=\tfrac{1}{720}\sqrt{X^4-898X^2+1920X+245761}$$
$$=[\,(X^2-449)/720\,,\,\overline{3(X-23)/4\,,\,(X+17)/60\,,\,-(X-15)/4\,,}$$
$$\overline{-(X-15)/4\,,\,(X+17)/60\,,\,3(X-23)/4\,,\,2(X^2-449)/720}\,].$$

Theorem. *There are no rational quartic polynomials $Y^2 = D(X)$ so that the continued fraction expansion of Y has period length nine, or eleven.*

Proof. For $t \in \mathbb{Q} \setminus \{0, \frac{1}{2}, 1\}$, the elliptic surface $\mathcal{C}_{10}(t)$ is given by

$$u_{10}(t) = -\frac{4t^6 - 16t^5 + 8t^4 + 8t^3 - 4t + 1}{4(t^2 - 3t + 1)^2},$$
$$v_{10}(t) = \frac{4t^3(t-1)(2t-1)}{(t^2 - 3t + 1)^2}, \qquad w_{10}(t) = \frac{2t^3 - 2t^2 - 2t + 1}{2(t^2 - 3t + 1)}, \tag{14}$$

with $\kappa_{10}(t) = -4t(t-1)(t^2 - 3t + 1)$;

The continued fraction expansion of $Y_{10}(X, t)$ is

$$\begin{aligned} Y = \Big[& X^2 - \tfrac{4t^6 - 16t^5 + 8t^4 + 8t^3 - 4t + 1}{4(t^2 - 3t + 1)^2}, \tfrac{(t^2-3t+1)^2}{2t^3(t-1)(2t-1)}\big(X - \tfrac{2t^3 - 2t^2 - 2t + 1}{2(t^2-3t+1)}\big), \\ & 2\big(X + \tfrac{2t^3 - 4t^2 + 4t - 1}{2(t^2 - 3t + 1)}\big), -\tfrac{t^2 - 3t + 1}{2t(t-1)(2t-1)}\big(X - \tfrac{2t^3 - 6t^2 + 4t - 1}{2(t^2 - 3t + 1)}\big), \\ & -\tfrac{2(t^2-3t+1)}{t}\big(X + \tfrac{2t-1}{2(t^2-3t+1)}\big), \tfrac{1}{2t^2(t-1)}\big(X + \tfrac{2t-1}{2(t^2-3t+1)}\big), \\ & \tfrac{2(t^2-3t+1)^2}{2t-1}\big(X - \tfrac{2t^3 - 6t^2 + 4t - 1}{2(t^2-3t+1)}\big), -\tfrac{1}{2t(t-1)(t^2-3t+1)}\big(X + \tfrac{2t^3 - 4t^2 + 4t - 1}{2(t^2 - 3t + 1)}\big), \\ & -\tfrac{2(t^2-3t+1)^3}{t^2(2t-1)}\big(X - \tfrac{2t^3 - 2t^2 - 2t + 1}{2(t^2-3t+1)}\big), \\ & -\tfrac{1}{2t(t-1)(t^2-3t+1)}\big(X^2 - \tfrac{4t^6 - 16t^5 + 8t^4 + 8t^3 - 4t + 1}{4(t^2-3t+1)^2}\big), \ldots \Big]. \end{aligned}$$

It follows from Proposition 4 that there is such an expansion with period length nine if and only if the equation $w^2 = \kappa_{10}(t)$ has a nontrivial solution in rationals t and w, that is, with $w \neq 0$. But there is no such solution.

We transform the equation by $t \mapsto 1/(t-1)$, $w \mapsto w/(t-1)^2$, yielding $w^2 = t^3 - 7t^2 + 15t - 10$, and note that the global minimal model of the cubic curve is $y^2 = x^3 - x^2 - x$. That is curve 80B2(A) in Cremona's tables [4], and we there read that the curve has rank 0 and its only rational point is the torsion point $(0, 0)$ of order 2. Hence there is no t such that $\kappa(t)$ is a square, except $t = 0$, 1, but those values give singular curves.

Just so, for $t \in \mathbb{Q} \setminus \{0, \frac{1}{2}, 1\}$, the elliptic surface $\mathcal{C}_{12}(t)$ is given by

$$u_{12}(t) = \frac{12t^8 - 120t^7 + 336t^6 - 468t^5 + 372t^4 - 168t^3 + 36t^2 - 1}{4(t-1)^6},$$
$$v_{12}(t) = \frac{4t(2t-1)(2t^2 - 2t + 1)(3t^2 - 3t + 1)}{(t-1)^4},$$
$$w_{12}(t) = \frac{6t^4 - 8t^3 + 2t^2 + 2t - 1}{2(t-1)^3}, \tag{15}$$

with $\kappa_{12}(t) = 4t(2t-1)^2(3t^2-3t+1)^3/(t-1)^{11}$; and

$$\begin{aligned}
Y_{12}(X,t) = [\,& X^2 + u_{12}(t)\,,\ \tfrac{(t-1)^4}{2t(2t-1)(2t^2-2t+1)(3t^2-3t+1)}\big(X - \tfrac{6t^4-8t^3+2t^2+2t-1}{2(t-1)^3}\big)\,,\\
& 2\big(X + \tfrac{6t^4-10t^3+8t^2-4t+1}{2(t-1)^3}\big)\,,\ -\tfrac{(t-1)^3}{2t(2t-1)(3t^2-3t+1)}\big(X - \tfrac{2t^4+2t^3-6t^2+4t-1}{2(t-1)^3}\big)\,,\\
& -\tfrac{2(3t^2-3t+1)}{(t-1)(2t^2-2t+1)}\big(X + \tfrac{2t^4-4t^3+6t^2-4t+1}{2(t-1)^3}\big)\,,\\
& \tfrac{(t-1)^6}{2t(2t-1)(3t^2-3t+1)^2}\big(X - \tfrac{(2t-1)(2t^2-2t+1)}{2(t-1)^3}\big)\,,\\
& \tfrac{2(2t-1)(3t^2-3t+1)}{(t-1)^5}\big(X - \tfrac{(2t-1)(2t^2-2t+1)}{2(t-1)^3}\big)\,,\\
& -\tfrac{(t-1)^{10}}{2t(2t-1)^2(2t^2-2t+1)(3t^2-3t+1)^2}\big(X + \tfrac{2t^4-4t^3+6t^2-4t+1}{2(t-1)^3}\big)\,,\\
& -\tfrac{2(2t-1)(3t^2-3t+1)^2}{(t-1)^8}\big(X - \tfrac{2t^4+2t^3-6t^2+4t-1}{2(t-1)^3}\big)\,,\\
& \tfrac{2(t-1)^{11}}{2t(2t-1)^2(3t^2-3t+1)^3}\big(X + \tfrac{6t^4-10t^3+8t^2-4t+1}{2(t-1)^3}\big)\,,\\
& \tfrac{2(2t-1)(3t^2-3t+1)^2}{(t-1)^7(2t^2-2t+1)}\big(X - \tfrac{6t^4-8t^3+2t^2+2t-1}{2(t-1)^3}\big)\,,\\
& \tfrac{(t-1)^{11}}{2t(2t-1)^2(3t^2-3t+1)^3}\big(X^2 + u_{12}(t)\big)\,,\ \ldots].
\end{aligned}$$

Much as before, when $m = 12$ we consider $t(t-1)(3t^2-3t+1) = w^2$ with $w \in \mathbb{Q}$, which expands to $w^2 = 3t^4 - 6t^3 + 4t^2 - t$. This quartic has a rational point $(1, 0)$. The transformation $t \mapsto -1/(t-1)$ and $w \mapsto w/(t-1)^2$ transforms the equation to $w^2 = t^3 + 7t^2 + 17t + 14$. Its global minimal model is $y^2 = x^3 + x^2 + x$, which is curve 48A4(A) of Cremona's tables [4]. That curve has rank 0 and its only rational point is the torsion point $(0, 0)$ of order 2. Hence there is no t such that $\kappa_{12}(t)$ is a square, except if $t = 0$, 1, which give singular curves. □

5 Pseudo-elliptic Integrals

Listing the fundamental unit in each of the function fields $\mathbb{Q}\big(Y_m(X,t)\big)$ is mere teratology[4] (teratology: the science or study of monstrosities ...), so we provide only examples. Note that to compute a unit $x(X) + y(X)Y$ one either computes the relevant convergent $x_{m-2}(X)/y_{m-2}(X)$ of the cited expansions or, more elegantly, one recalls that the unit is the product of the complete quotients $(Y + P_h)/Q_h$ for $h = 1, \ldots\ m-1$.

The following is a list of example pseudo-elliptic integrals, see [11],

$$\int \frac{f_m(z)\,dz}{\sqrt{D_m(z)}} = \log\big(x_m(z) + y_m(z)\sqrt{D_m(z)}\,\big).$$

In each case the reader might verify that indeed $x' = fy$ and $x^2 - Dy^2$ is constant.

[4] The truly interested reader will learn more from computing them for herself than from studying a list — in any case, the length and complexity of such a list would have forced me to exceed my page limit.

$$x_{12}(z) = z^{12} - 118z^{11} + 16028z^{10} - 1069154z^9 + 72544053z^8 - 2910120156z^7$$
$$+115293384192z^6 - 2435904763524z^5 + 49959577428123z^4 - 3156443198606z^3$$
$$-6523744685908252z^2 + 264671040329753798z - 1519185098148240209;$$
$$y_{12}(z) = z^{10} - 118z^9 + 14517z^8 - 944616z^7 + 57651426z^6 - 2264475780z^5$$
$$+79914037266z^4 - 1800781684584z^3 + 34360879041117z^2$$
$$-338671088037302z + 2242974918048761;$$
$$f_{12}(z) = 12z + 118; \quad D_{12}(z) = (z^2 + 1511)^2 + 107520(z - 13).$$

$$x_{10}(z) = z^{10} - 125z^8 - 1600z^7 + 7450z^6 + 128000z^5 + 457750z^4$$
$$-4504000z^3 - 22308875z^2 + 274924375;$$
$$y_{10}(z) = z^8 - 100z^6 - 1120z^5 + 4470z^4 + 64000z^3$$
$$+183100z^2 - 1351200z - 4461775;$$
$$f_{10}(z) = 10z; \quad D_{10}(z) = (z^2 - 25)^2 - 960(z - 1).$$

$$x_9(z) = z^9 - 9z^8 - 108z^7 + 828z^6 + 5454z^5 - 29646z^4 - 131868z^3$$
$$+467532z^2 + 1190457z - 3028401;$$
$$y_9(z) = z^7 - 9z^6 - 75z^5 + 627z^4 + 2403z^3 - 15579z^2 - 28377z + 132273;$$
$$f_9(z) = 9(z + 1); \quad D_9(z) = (z^2 - 33)^2 - 192(z + 3).$$

$$x_8(z) = z^8 - 10z^7 - 50z^6 + 1006z^5 - 976z^4 - 34526z^3$$
$$+108946z^2 + 413690z - 1829009;$$
$$y_8(z) = z^6 - 10z^5 - 25z^4 + 660z^3 - 1313z^2 - 11306z + 41369;$$
$$f_8(z) = 8z + 10; \quad D_8(z) = (z^2 - 25)^2 + 192(z + 7).$$

$$x_7(z) = z^7 + z^6 - 31z^5 - 103z^4 + 331z^3 + 1435z^2 - 429z - 5557;$$
$$y_7(z) = z^5 + z^4 - 22z^3 - 62z^2 + 133z + 429;$$
$$f_7(z) = 7z - 1; \quad D_7(z) = (z^2 - 9)^2 - 64(z - 1).$$

$$x_6(z) = z^6 - 2z^5 + 8z^4 - 4z^3 + 8z^2 + 8z;$$
$$y_6(z) = z^4 - 2z^3 + 6z^2 - 4z + 4;$$
$$f_6(z) = 6z + 2.; \quad D_6(z) = (z^2 + 2)^2 + 8z.$$

$$x_5(z) = z^5 - z^4 + 3z^3 + z^2 + 2; \quad y_5(z) = z^3 - z^2 + 2z;$$
$$f_5(z) = 5z + 1; \quad D_5(z) = (z^2 + 1)^2 + 4z.$$

$$x_4(z) = z^4 - 2z^3 + 2z^2 + 4z - 4; \quad y_4(z) = z^2 - 2z + 2;$$
$$f_4(z) = 4z + 2; \quad D_4(z) = z^4 + 8(z + 1).$$

$$\int \frac{(3z - s)\,dz}{\sqrt{(z^2 - s^2)^2 + t(z - s)}}$$
$$= \log\bigl(1 + 2(z + s)(z^2 - s^2)/t + 2\bigl((z + s)/t\bigr)\sqrt{(z^2 - s^2)^2 + t(z - s)}\,\bigr).$$

$$\int \frac{2z\,dz}{\sqrt{(z^2 + s)^2 + t}} = \log\bigl(z^2 + s + \sqrt{(z^2 + s)^2 + t}\,\bigr).$$

One readily recognises the final, $m = 2$, example as an elementary integral by setting $w = z^2 + s$. That might make one wonder whether there are rational transformations that nakedly reveal the elementary nature of the integrals in each case. The answer is, of course, yes; a helpful reference is [14], pp38*ff*.

6 Remark

The attentive reader will have noticed an unexpected feature of the detailed continued fraction expansions for m at least 4. In each case the third partial quotient, a_2 , is of the shape $2(X - c)$, moreover with $c = -w + 1$. Of course, such an observation may well be no more than an artefact of Kubert's parametrisations on which ours are based. Indeed, the curves on page 106 depend on just two[5] parameters, there called b and c, so our three parameters cannot be independent. Specifically, they happen all to satisfy the identity $4(u_m + w_m^2) = v_m$. Although u and w^2 do have the same weight, that weight is different from the weight of v, so that coefficient 2 *is* artificial. A normalisation ($xX \mapsto X$, $x^2Y \mapsto Y$, so $u' = u/x^2$, $v' = v/x^3$, $w' = w/x$) changes the identity to $4(u'^2 + w') = xv'$, and the 2 to $2/x$.

References

1. William W. Adams and Michael J. Razar, 'Multiples of points on elliptic curves and continued fractions', *Proc. London Math. Soc.* **41** (1980), 481–498.
2. T. G. Berry, 'On periodicity of continued fractions in hyperelliptic function fields', *Arch. Math.* **55** (1990), 259–266.
3. Henri Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Mathematics **138** (New York: Springer–Verlag, 1993).
4. J. E. Cremona, *Algorithms for Modular Elliptic Curves*, 2nd edition, Cambridge University Press, 1997.
5. Everett W. Howe, Franck Leprévost, and Bjorn Poonen, 'Large torsion subgroups of split Jacobians of curves of genus two or three', *Forum Math.* **12**.3 (2000), 315–364 (MR2001e:11071).

[5] *Of course* an elliptic curve depends on just two parameters, say the *two* Eisenstein series G_4 and G_6 .

6. Daniel Sion Kubert, ‘Universal bounds on the torsion of elliptic curves’, *Proc. London Math. Soc.* **33**.3 (1976), 193–237.
7. B. Mazur, ‘Modular curves and the Eisenstein ideal’, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 33–186.
8. Abderrahmane Nitaj, ‘Détermination de courbes elliptiques pour la conjecture de Szpiro’, *Acta Arith.* **85**.4 (1998), 351–376.
9. Abderrahmane Nitaj, ‘Isogènes des courbes elliptiques définies sur les rationnels’, to appear in *J. Combinatorial Math.*
10. Oskar Perron, *Die Lehre von den Kettenbrüchen*, 2nd edition, 1929 (Chelsea Publishing Company, New York, N Y).
11. Alfred J. van der Poorten and Xuan Chuong Tran, ‘Quasi-elliptic integrals and periodic continued fractions’, *Monatshefte Math.*, 131 (2000), 155-169.
12. Alfred J. van der Poorten, ‘Non-periodic continued fractions in hyperelliptic function fields’, (Dedicated to George Szekeres on his 90th birthday), *Bull. Austral. Math. Soc.* **64** (2001), 331–343.
13. A. J. van der Poorten and H. C. Williams, ‘On certain continued fraction expansions of fixed period length’, *Acta Arith.* **89**.1 (1999), 23–35 (MR2000m:11010).
14. Viktor Prasolov and Yuri Solovyev, *Elliptic functions and elliptic integrals*, translated from the Russian manuscript by D. Leites, Translations of Mathematical Monographs, 170. American Mathematical Society, Providence, RI, 1997; x+185 pp.
15. A. Schinzel, ‘On some problems of the arithmetical theory of continued fractions’, *Acta Arith.* **6** (1961), 393–413; and *ibid.* **7** (1962), 287–298.
16. Wolfgang M. Schmidt, ’On continued fractions and diophantine approximation in power series fields’, *Acta Arith.* **95** (2000), 139–166.
17. Jing Yu, ‘Arithmetic of hyperelliptic curves’, manuscript marked *Aspects of Mathematics*, Hong Kong University, 1999; see pp4–6.