REFERENCES

1. R. M.Young, Euler's constant, *Math. Gazette* **75** (1991) 187–190.
2. D. W. DeTemple, A quicker convergence to Euler's constant, *Amer. Math. Monthly* **100** (1993) 468–470.
3. P. L. Rippon, Convergence with pictures, *Amer. Math. Monthly* **93** (1986) 476–478.
4. D. W. DeTemple and S. H. Wang, Half integer approximations for the partial sums of the harmonic series, *J. Math. Anal. Appl.* **160** (1991) 149–156.

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Life on the Edge

Alf van der Poorten

1. INTRODUCTION. One knows that $\log(1-z) = -\sum_{n=1}^{\infty} z^n/n$ for $|z| \le 1$ and $z \ne 1$. Because $1 - e^{i\theta} = -2i\sin(\theta/2)\cdot e^{i\theta/2}$ and $-i = e^{-\pi i/2}$, we see that

$$\log(1-e^{i\theta}) = \log(-i) + \log\left(2\sin\frac{\theta}{2}\right) + \log e^{i\theta/2} = \log\left(2\sin\frac{\theta}{2}\right) - i\left(\frac{\pi}{2}-\theta\right),$$

and on taking real and imaginary parts of $-\log(1-z)$ with $z = e^{i\theta} = \cos\theta + i\sin\theta$, it follows that

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n} = -\log\left(2\sin\frac{\theta}{2}\right)$$

and

$$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n} = \frac{\pi}{2} - \frac{\theta}{2} \tag{1}$$

for $0 < \theta < 2\pi$.

The relevant rule of thumb is that power series can safely be treated as if they were polynomials of [very] high degree *provided one stays well away from the boundary of the disc of convergence*. So, guessing that $\log(1-e^{i\theta})$ has imaginary part $\sum_{n\ge1}(\sin n\theta)/n = (\pi-\theta)/2$ for $0 < \theta < 2\pi$ is scary stuff requiring the presence of a qualified mathematician.* Do not try it at home.

In fact, oops! What if θ creeps down to zero? Surely, all the terms of the series become zero? But its purported sum becomes $\pi/2$!

2. EVALUATION OF AN INTEGRAL. Not to worry. Look carefully at the graph of $(\sin x)/x$. It's the sine curve wriggling pathetically as it is squeezed between the hyperbolae $xy = 1$ and $xy = -1$.

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*MGR: Mathematical Guidance Recommended. Possible use of strong technical language and presence of naked singularities.

I suggest that it is just the comparison of a sum with an integral that, as $\theta \to 0$, one has

$$\int_0^\infty \frac{\sin \theta x}{x}\,dx - \sum_{n\geq 1} \frac{\sin n\theta}{n} \longrightarrow 0. \tag{2}$$

Call the integral I_θ. So, by (2), we have $\lim_{\theta\to 0} I_\theta = \pi/2$. However, it happens to be well known (see the acknowledgement at the end of this note) that

$$I_1 = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{3}$$

Indeed, *mirabile dictu*,[†] the integral I_θ is independent of θ.

Normally, I'd feel sufficiently frightened and confused by all this to call for an *analyst* (that's the official term for mathematicians who live on the boundary and criticise people who wander there inadvertently). However, the series $1/(1-z) = \sum_{n\geq 0} z^n$ obviously is perfectly well behaved on its boundary, other than for its pole at $z = 1$, whence its integral $\log(1-z) = -\sum_{n\geq 1} z^n/n$ is also well behaved on *its* boundary away from $z = 1$. Thus the argument just now sketched *proves* that $I = I_1$ in fact is $\pi/2$, so that (3) is now also well known to us.

Mind you, I had better detail my main argument. What I had in mind was the remark

$$\int_0^N \frac{\sin \theta x}{x}\,dx \approx \sum_{k=1}^{nN} \frac{\sin \theta(k/n)}{k/n} \cdot \frac{1}{n} = \sum_{k=1}^{nN} \frac{\sin k(\theta/n)}{k}.$$

Now let $N \to \infty$, and also $n \to \infty$. But, while that's more than good enough for a "suggestion," it leaves me—and my analyst colleagues—a little uncomfortable.

So, let's try again, having first observed that both our integral and our sum, after collecting their terms into groups of the same sign, may be viewed as alternating sums with monotonically decreasing terms. It was therefore always clear that both converged, albeit only conditionally. For a reasonable function f, we have

$$\int_n^{n+1} f(x)\,dx = \tfrac{1}{2}(f(n+1) + f(n)) - \tfrac{1}{12} f''(\xi)$$

for some ξ in $[n, n+1]$, by the trapezoidal rule. Now sum the cases $n = 0, 1, 2, \ldots$. For the example $f(x) = (\sin \theta x)/x$, we have $f(0) = \theta$; the second derivative of f is $-\theta^2(\sin\theta x)/x$ plus two less problematic functions, $-\theta(\cos\theta x)/x^2$ and $2(\sin\theta x)/x^3$. In effect as just remarked, we know that the sum $\theta^2 \sum_{n=1}^{\infty} (\sin \theta\xi_n)/\xi_n$, where ξ_n belongs to $[n, n+1]$, converges conditionally, and each of the other two sums converges, so it is plain that when $\theta \to 0$ we have the suggestion (2). This more defensible argument is just a junior version of Euler-Maclaurin summation. For it, and the real thing, see [**1**].

3. A RELATED QUESTION. Recall that $\sum_{n\geq 1}(\sin n\theta)/n = (\pi - \theta)/2$ for $0 < \theta < 2\pi$. Carefully integrate with respect to θ and hence evaluate the sum $\sum_{n\geq 1} 1/n^2$ ("carefully" means not forgetting that indefinite integration leaves an undetermined constant). On having successfully obtained $\zeta(2) = \sum_{n\geq 1} 1/n^2 = \pi^2/6$, one can now try to be Euler by integrating twice more to obtain $\zeta(4)$, then twice more to .... Can this be done by mere mortals?

[†] *mirabile dictu*: wonderful it is to relate.

Answer. Yup. It sure can. We shall suppose that we may integrate term by term (as indeed we may), thus getting

$$\sum_{n\geq 1}\int_0^{\pi}\frac{\sin n\theta}{n}\,d\theta=\int_0^{\pi}\left(\frac{\pi}{2}-\frac{\theta}{2}\right)d\theta=\left[\frac{\pi\theta}{2}-\frac{\theta^2}{4}\right]_0^{\pi}=\frac{\pi^2}{4}.$$

On the left-hand side we obtain $\left[\sum_{n\geq 1}-(\cos n\theta)/n^2\right]_0^{\pi}$. It is

$$\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots\right)+\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right),$$

which we recognise as $2(1-2^{-2})\zeta(2)=3\zeta(2)/2$. Thus $\zeta(2)=(2/3)(\pi^2/4)=\pi^2/6$. Note that here, and from here on, the series do converge absolutely so that we can safely reorder terms arbitrarily.

To continue conveniently, we do need the indefinite integral, yielding

$$-\sum_{n\geq 1}\frac{\cos n\theta}{n^2}=\frac{\pi\theta}{2}-\frac{\theta^2}{4}-\frac{\pi^2}{6}.$$

Now integrating once more, we see that the next integration constant is 0, so that integrating yet one more time provides

$$\begin{aligned}\sum_{n\geq 1}\int_0^{\pi}\frac{-\sin n\theta}{n^3}\,d\theta&=\int_0^{\pi}\left(\frac{\pi\theta^2}{4}-\frac{\theta^3}{12}-\frac{\pi^2\theta}{6}\right)d\theta\\&=\left[\frac{\pi\theta^3}{12}-\frac{\theta^4}{48}-\frac{\pi^2\theta^2}{12}\right]_0^{\pi}=-\frac{\pi^4}{48}.\end{aligned}$$

Here we have $-2(1-2^{-4})\zeta(4)=-15\zeta(4)/8$ on the left, so $\zeta(4)=\pi^4/90$. It is the integration constant so, ... Two integrations further on we evaluate

$$\left[\frac{\pi\theta^5}{240}-\frac{\theta^6}{1440}-\frac{\pi^2\theta^4}{144}+\frac{\pi^4\theta^2}{180}\right]_0^{\pi}=\frac{\pi^6}{480}.$$

On the left we have $2(1-2^{-6})\zeta(6)=63\zeta(6)/32$, so $\zeta(6)=\pi^6/945$; and so on.

Euler knew by different methods that

$$\zeta(2k)=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+\frac{1}{4^{2k}}+\frac{1}{5^{2k}}+\cdots\cdots=(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}.$$

It is at first surprising that the present process provides a sequential evaluation of the Bernoulli numbers B_{2k}. Recall, however, that $z/(e^z-1)=\sum_{n\geq 0}B_n z^n/n!$ generates the Bernoulli numbers; we have tortured $\log(1-e^{i\theta})$.

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REFERENCES

1. R. P. Boas, Jr., Partial sums of infinite series, and how they grow, *Amer. Math. Monthly* **84** (1977) 237–258; see Chapter 2.
2. R. Courant, *Differential and Integral Calculus*, Blackie & Son Limited, London and Glasgow, 1937.
3. Ira Rosenholtz, Tangent sequences, world records, π, and the meaning of life: some applications of number theory to calculus, *Math. Mag.* **72** (1999) 367–376.

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Two Conjectures Concerning Sums of Squares

It is well known that every positive integer is the sum of *at most* four positive squares. We state two conjectures related to this fact.

Conjecture 1. A positive integer is the sum of *precisely* four positive squares if and only if it is not 1, 3, 5, 9, 11, 17, 29, or 41 nor of one of the forms $2 \times 4^\alpha$, $6 \times 4^\alpha$, or $14 \times 4^\alpha$.

Conjecture 2. A positive integer is the sum of at most four *distinct* positive squares if and only if it does not have one of the following forms: $2 \times 4^\alpha$, $3 \times 4^\alpha$, $6 \times 4^\alpha$, $7 \times 4^\alpha$, $11 \times 4^\alpha$, $15 \times 4^\alpha$, $18 \times 4^\alpha$, $19 \times 4^\alpha$, $22 \times 4^\alpha$, $23 \times 4^\alpha$, $27 \times 4^\alpha$, $31 \times 4^\alpha$, $33 \times 4^\alpha$, $43 \times 4^\alpha$, $47 \times 4^\alpha$, $55 \times 4^\alpha$, $67 \times 4^\alpha$, $103 \times 4^\alpha$.

We prove the easy “only if” parts of each conjecture.

Conjecture 1. It is easy to check that none of 1, 2, 3, 5, 6, 9, 11, 14, 17, 29, or 41 is the sum of four positive squares. To complete a proof by descent, we observe that if n is even and $4n$ is the sum of four positive squares, then n is likewise the sum of four positive squares.

Conjecture 2. That none of 2, 3, 6, 7, 11, 12, 15, 18, 19, 22, 23, 27, 28, 31, 33, 43, 44, 47, 55, 60, 67, 76, 92, 103, 108, 124, 132, 172, 188, 220, 268, or 412 is the sum of at most four distinct squares can be verified directly. To complete a proof by descent, we observe that if n is even and $4n$ is the sum of at most four distinct squares, then so is n.

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