Squares from products of integers

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1 Introduction

It is well known that the product of any four consecutive integers differs by one from a perfect square. However there is no integer n, other than four, so that the product of any n consecutive integers always differs from a perfect square by some fixed integer $c = c(n)$ depending only on n.

The argument [7] showing this relies on the fact that a polynomial taking too many square values must be the square of a polynomial (see [6, Chapter VIII.114 and .190], and [2]). One might therefore ask whether there are polynomials, other than integer multiples of $x(x+1)(x+2)(x+3)$ and $4x(x+1)$, that have integer zeros and differ by a nonzero constant from the square of a polynomial. We will show that this is quite a good question in that it has a nontrivial answer, inter alia giving new insight into the results of [7]. As an example of the phenomenon, the reader might check that

$$1\cdot2\cdot3\cdot5\cdot6\cdot7+36=4^2\cdot9^2 \qquad 2\cdot3\cdot4\cdot6\cdot7\cdot8+36=5^2\cdot18^2$$
$$3\cdot4\cdot5\cdot7\cdot8\cdot9+36=6^2\cdot29^2 \qquad 4\cdot5\cdot6\cdot8\cdot9\cdot10+36=7^2\cdot42^2 \qquad \ldots .$$

2 Squares from products of a set of integers

We study polynomials $P_S(x) = \prod_{s\in S}(x+s)$ and find all nonempty sets S of integers with the property that for some rational number c, $P_S + c$ is the square of a polynomial.

Call that polynomial $a(x) = a_{S,c}(x)$. Then we have

$$P = a^2 - c = (a+\sqrt{c}\,)(a-\sqrt{c}\,).$$

It follows there is a partition $S = R \cup T$ of S so that

$$a(x)+\sqrt{c} = \prod\nolimits_{r\in R}(x+r) \quad \text{and} \quad a(x)-\sqrt{c} = \prod\nolimits_{t\in T}(x+t). \tag{1}$$

Because $S \subset \mathbb{Z}$, it follows that $c = k^2$ for some rational k.

Since $a(x)+\sqrt{c}$ and $a(x)-\sqrt{c}$ have the same degree, we see that R and T have the same cardinality, m say, and S has cardinality $2m$. Because the polynomials $a(x)\pm\sqrt{c}$ differ by a constant, it follows that the respective elementary symmetric functions in the integers $r \in R$ and the integers $t \in T$, other than those of order m, coincide. Equivalently, but more strikingly, we have for $j = 0, 1, \ldots, m-1$, the identity

$$\sum\nolimits_{r\in R} r^j = \sum\nolimits_{t\in T} t^j\,. \tag{2}$$

This follows immediately from Newton's formulas whereby if

$$f(x) := (x-x_1)(x-x_2)\cdots(x-x_n) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_{n-1}x + \sigma_n\,,$$

then for $h = 0, 1, 2, \ldots$

$$s_h\sigma_0 + s_{h-1\sigma_1} + \cdots + s_{h-n+1}\sigma_{n-1} + s_{h-n}\sigma_n = 0\,, \tag{3}$$

where the s_j are the power sums $x_1^j + x_2^j + \cdots + x_n^j$ and, of course, $\sigma_0 = 1$ while $s_k = 0$ for $k < 0$; and one replaces s_0 by h.

In brief, we have $s_1 = -\sigma_1$, $s_2 = -\sigma_1 s_1 - 2\sigma_2$, ... illustrating that if two polynomials

$$x^n + \sigma_1 x^{n-1} + \cdots \sigma_{n-1} x + \sigma_n \quad \text{and} \quad x^n + \sigma_1' x^{n-1} + \cdots \sigma_{n-1}' x + \sigma_n'$$

coincide, other than perhaps for their constant coefficients, then we have $s_0 = s_0'$, $s_1 = s_1'$, ..., $s_{n-1} = s_{n-1}'$ for the power sums in their respective zeros; whence (2).

Moreover, one sees that the case $m = 1$ is trivial, and the case $m = 2$ is nearly trivial. Indeed, for $m = 1$ the conditions (2) are essentially empty, and for $m = 2$ it is plain that one may select any three of the integers r_1, r_2, t_1, t_2, and obtain an integer for the fourth; in that case, incidentally, one has $c = (r_1 r_2 - t_1 t_2)^2/4$.

A minor digression

Of course Newton's formulas are well known. In order to recall them well it may be useful to oberve that (3) is the remark, here with $s_0 = n$, that plainly

$$\frac{f'(x)}{f(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \cdots \frac{1}{x - x_n} = \sum_{m=0}^{\infty} s_m x^{-m-1} .$$

Now multiply by $f(x)$ and compare coefficients on the two sides.

3 The Prouhet–Tarry–Escott problem

Seeing (2), one recalls that the Tarry–Escott problem is precisely the issue of finding distinct sets of integers r_1, r_2, ..., r_n and t_1, t_2, ..., t_n with

$$r_1^j + r_2^j + \cdots + r_n^j = t_1^j + t_2^j + \cdots + t_n^j$$

for $j = 0, 1, 2, \ldots, j = m$. A solution is said to be *ideal* if $m = n - 1$. The critical reference is the observation by Wright [9] that the question of Tarry and Escott had already been dealt with by Prouhet [8].

Clearly, our remarks above amount to the following theorem.

Theorem *Let S be a finite set of integers and set $P_S(x) = \prod_{s\in S}(x + s)$. Then P_S differs by a constant c from the square of a polynomial if and only if S is the disjoint union of sets R and T that provide an ideal solution to the Tarry–Escott problem.*

Thus [7] reminds us that there are no ideal solutions $R \cup T = S$ to the Tarry–Escott problem for which S is an arithmetic progression of more than four integers.

There is activity in the matter of finding new solutions to the Tarry–Escott problem; it is best followed on the web, starting from [1] or [5]. The following sporadic examples come from there and other linked sources.

The opening example

$$x(x + 1)(x + 2)(x + 4)(x + 5)(x + 6) + 36 = (x + 3)^2(x^2 + 6x + 2)^2.$$

From Tarry's ideal symmetric solution of 1912

$$x(x+1)(x+2)(x+5)(x+6)(x+10)(x+12)(x+16)(x+17)(x+20)(x+21)(x+22)+2540160000$$
$$=(x^6+66x^5+1633x^4+18612x^3+95764x^2+179520x+50400)^2.$$

From Escott's ideal symmetric solution of 1910

$$x(x+1)(x+13)(x+18)(x+27)(x+38)(x+44)(x+58)(x+64)(x+75)(x+84)(x+89)(x+101)(x+102)+c$$
$$=(x+51)^2(x^6+306x^5+34801x^4+1793364x^3+40430980x^2+315284448x+136936800)^2;$$

of course, here $c = (51 \cdot 136936800)^2 = 6983776800^2$.

Shifting by primes

$$(x+7)(x+11)(x+13)(x+19)(x+29)(x+31)+82944=(x^3+55x^2+887x+4145)^2\,;$$
$$(x+11)(x+13)(x+19)(x+23)(x+29)(x+31)+25600=(x+21)^2(x^2+42x+357)^2.$$

Shifting by squares

$$(x+1^2)(x+5^2)(x+6^2)(x+9^2)(x+10^2)(x+11^2)+50400^2=(x^3+182x^2+8281x+58500)^2.$$

Acknowledgement

We were quite put out when told by an apparently omniscient adviser that the connection with the Prouhet–Tarry–Escott problem is well-known; it appears in a 1935 paper [4]. In truth, we might equally have been criticised for not reading Dickson's *History of the Theory of Numbers* [3] more carefully.

The present remarks arise from an incidental talk given by one of us at the Annual Meeting of the Australian Mathematical Society at the Australian National University, September 2001, an interesting question asked by the other of us, and a helpful observation of Igor Shparlinski.

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