Part A

(Please use a separate book for Part A.)

Question 1. [**20 marks**]

Consider the wave equation

$$u_{tt} = c^2 u_{xx} \tag{1}$$

on the unbounded domain $-\infty < x < \infty$, where c is a positive constant.

(a) Derive the general solution of the wave equation (1) and show that it is a sum of two travelling waves: $u(x,t) = F(x - ct) + G(x + ct)$.

(b) Find the solution of the wave equation (1) that satisfies the initial conditions

$$u\left(x,0\right) = \varphi\left(x\right), \quad u_t\left(x,0\right) = g(x) \quad \text{on} \quad -\infty < x < \infty,$$

where φ and g are given functions.

(c) Find the solution w of the wave equation

$$w_{tt} = c^2 w_{xx} \tag{2}$$

on the half-line $0 < x < \infty$ and $t \geq 0$, that satisfies the initial conditions

$$w\left(x,0\right) = \varphi\left(x\right), \quad w_t\left(x,0\right) = 0 \quad \text{for } x \geq 0, \tag{3}$$

and the Dirichlet boundary condition

$$w\left(0,t\right) = 0 \quad \text{for all } t. \tag{4}$$

(d) Find the solution v of the wave equation

$$v_{tt} = c^2 v_{xx} \tag{5}$$

on the half-line $0 < x < \infty$ and $t \geq 0$, that satisfies the initial conditions

$$v\left(x,0\right) = e^{-(x-5)^2}, \quad v_t\left(x,0\right) = 0 \quad \text{for } x \geq 0,$$

and the Neumann boundary condition

$$v_x\left(0,t\right) = 0 \quad \text{for all } t.$$

Question 2. [**20 marks**]

Use the method of separation of variables to solve Helmholtz equation

$$\Delta\varphi + k^2\varphi = 0$$

for the function $\varphi(x, y, z)$ in the cubic region $0 < x < 2$, $0 < y < 2, 0 < z < 2$ with the 'hard' boundary condition: $\frac{\partial\varphi}{\partial n} = 0$ on the boundary, where $\mathbf{n}$ is the outward unit normal vector. What condition must k satisfy in order to have non-trivial solutions? How many solutions are there when $0 < k < 1$?

Question 3. [**20 marks**]

(a) Show that the Fourier cosine series of the function $f(x) = x(\pi - x)$ defined for $0 < x < \pi$ is

$$\frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{[1 + (-1)^n]}{n^2} \cos nx.$$

(b) Find the standing wave solutions $u(x, t)$ of the homogeneous wave equation

$$u_{tt} = u_{xx}$$

in the region $0 < x < \pi$ and $t > 0$, that satisfies the Neumann boundary conditions

$$u_x(0, t) = u_x(\pi, t) = 0,$$

and the initial conditions

$$u(x, 0) = x(\pi - x), u_t(x, 0) = \cos x$$

You may use the result in part (a).

(c) Consider vibrations of a finite string with free ends, modelled by the initial boundary value problem in part (b). Use the solution in part (b) to sketch the graph of the *first two modes* of vibration of the string at the time instants where t takes values $0, \pi/2, \pi$.

Question 4. [**20 marks**]

Consider the cylinder of radius a with its axis equal to the z-axis. The interior of the cylinder with a rigid surface forms an acoustic waveguide. Let C denote the cross-section of the cylinder, including the boundary and interior of the circle, in the plane $z = 0$; let ∂C denote its boundary. Let ϕ be an acoustic velocity potential, satisfying the wave equation and having form

$$\phi = \phi(x, y, z, t) = e^{i(k_z z - \omega t)}\Psi(x, y), \tag{6}$$

where Ψ is a function only of x and y (independent of z); ω and k_z are given constants.

(a) What sort of wave does the form (6) represent? Show that Ψ satisfies

$$\nabla^2\Psi + \left(k^2 - k_z^2\right)\Psi = 0$$

where $k = \omega/c$, with c denoting the plane wave speed in the unbounded medium.

(b) Use the identity

$$\nabla \cdot (\Psi\nabla\Psi) = |\nabla\Psi|^2 + \Psi\nabla^2\Psi$$

and the divergence theorem to show that

$$\int_{\partial C} \Psi\frac{\partial\Psi}{\partial r}ds = \iint_C |\nabla\Psi|^2\, dS + \left(k_z^2 - k^2\right)\iint_C \Psi^2 dS, \tag{7}$$

where $\frac{\partial\Psi}{\partial r}$ denotes the radially outward derivative on ∂C.

(c) Explain why the left hand side of (7) is zero. Deduce that if $k_z > k > 0$, then Ψ must be the zero function.

(d) Suppose that, in standard cylindrical coordinates (r, θ, z), the function Ψ is independent of θ, so that we may write $\Psi = \Psi(r)$. Show that Ψ satisfies the ordinary differential equation

$$\frac{d^2}{dr^2}\Psi + \frac{1}{r}\frac{d}{dr}\Psi + \left(k^2 - k_z^2\right)\Psi = 0, \tag{8}$$

subject to the boundary condition $\Psi'(a) = 0$. [You may assume that the formula

$$\Delta\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}.]$$

(e) Now suppose $k > k_z > 0$. Show that the equation (8) has solutions that are bounded in the interior of the cylinder only when $(k^2 - k_z^2)^{\frac{1}{2}}\, a$ is a zero of J_0', where J_0 denotes the Bessel function of first kind and order zero.

Let $\nu_0 < \nu_1 < \nu_2 < \nu_3 < \ldots$ denote the non-negative zeros of J_0' arranged in ascending order; $\nu_0 = 0$. Deduce that

$$k_z = k_m = \sqrt{k^2 - \frac{\nu_m^2}{a^2}}, \tag{9}$$

where $m = 0, 1, 2, 3, \ldots$. What is the speed of propagation of the mode corresponding to the choice of k_m as given by (9)?

[The differential equation satisfied by Bessel functions of order ν is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \nu^2\right)y = 0.]$$

Part B

(Please use a separate book for Part B.)

Question 5. [**20 marks**]

(a) Explain what is meant by a travelling wave solution to a partial differential equation for the function $u = u(x,t)$. Find all travelling wave solutions of the equation

$$u_{tt} = u_{xx} + u_{xxxx} \tag{10}$$

which have speed c satisfying $0 < c < 1$.

(b) By considering wave-train solutions, determine the dispersion relation of the equation (10). Is the equation dispersive?

(c) Consider the three-dimensional wave equation for the function $\varphi = \varphi(\mathbf{x}, t)$ where $\mathbf{x}$ denotes the point (x, y, z) in $\mathbf{R}^3$ and t denotes time:

$$\frac{1}{c^2}\varphi_{tt} = \Delta\varphi, \tag{11}$$

where c is a positive constant.

(i) Let f be a function of one real variable that is twice differentiable, and let $\mathbf{n}$ denote a unit vector in $\mathbf{R}^3$. Show that the function

$$\varphi_1(\mathbf{x}, t) = f(\mathbf{n} \cdot \mathbf{x} - ct)$$

satisfies the wave equation (11). What type of wave does this function represent? What is its speed and direction?

(ii) Let $r = |\mathbf{x}|$. By considering the function $v = r\varphi$, find the most general form of solution to the wave equation (11) that is spherically symmetric about the origin; show that it represents the sum of two waves, and describe their speeds and directions.

(iii) Let $\psi = \psi(\mathbf{x}, t)$ represent an outgoing, spherically symmetric wave that satisfies the condition that $r\psi$ is bounded in $\mathbf{R}^3$. Show that

$$r\left(\frac{\partial\psi}{\partial r} + \frac{1}{c}\frac{\partial\psi}{\partial t}\right) \to 0, \text{ as } r \to \infty.$$

What condition does the corresponding inwardly travelling spherically symmetric wave satisfy?

[You may assume that in standard spherical polar coordinates (r, θ, ϕ) the Laplacian takes the form

$$\Delta\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.]$$

Question 6. [**20 marks**]

(a) (i) Show that $\operatorname{sech}^2 x + \tanh^2 x = 1$ for all real x.

(ii) Find the derivative of the function $f(x) = \operatorname{sech} x$; find the derivative of its *inverse* function $f^{-1}(x) = \operatorname{sech}^{-1} x$. Deduce that, if a is a positive constant,

$$\int \frac{dx}{x\sqrt{a-x^4}} = -\frac{1}{2\sqrt{a}} \operatorname{sech}^{-1}\left(\frac{x^2}{\sqrt{a}}\right) + C,$$

where C denotes an arbitrary constant.

(b) Consider the following partial differential equation for the function $u = u(x,t)$:

$$u_t + 15u^4 u_x + u_{xxx} = 0.$$

Let $u = f(x - ct)$ be a travelling wave solution of this equation satisfying the constraint that $f(z), f'(z)$ and $f''(z) \to 0$ as $z \to \pm\infty$.

(i) Show that f satisfies the differential equation $(f')^2 = cf^2 - f^6$.

(ii) Use the result of part (a) to determine the solution of this differential equation.

(iii) How is the amplitude (maximum height) of the travelling wave solution related to its speed?

(iv) Sketch the waveform of the travelling wave solution for the value $c = 1$.

Question 7. [**20 marks**]

Let Ω be a bounded region in $\mathbf{R}^3$, with a smooth surface $\partial\Omega$; let k denote a positive number. For any points $\mathbf{x}$ and $\mathbf{x}_0$ in Ω, define the function $G(\mathbf{x};\mathbf{x}_0)$ by

$$G(\mathbf{x};\mathbf{x}_0) = -\frac{1}{4\pi r} e^{-ikr}, \text{ where } r = |\mathbf{x} - \mathbf{x}_0|.$$

(a) What type of wave does the function $G(\mathbf{x};\mathbf{x}_0)e^{i\omega t}$ represent (ω a fixed positive number)?

(b) Show that, for each fixed $\mathbf{x}_0$ in Ω, the function $G(\mathbf{x};\mathbf{x}_0)$ satisfies the Helmholtz equation

$$\nabla^2 G(\mathbf{x};\mathbf{x}_0) + k^2 G(\mathbf{x};\mathbf{x}_0) = 0$$

at each point $\mathbf{x}$ in Ω, except $\mathbf{x} = \mathbf{x}_0$. (The differentiation is done with respect to the variable denoted by $\mathbf{x}$; you may assume the form of the Laplacian in spherical coordinates as stated at the end of question 5)

(c) Let $f(\mathbf{x})$ be a given continuous function defined in the region Ω. Suppose that $\varphi = \varphi(\mathbf{x})$ is a solution of the *inhomogeneous* Helmholtz equation at all points in Ω:

$$\Delta\varphi(\mathbf{x}) + k^2\varphi(\mathbf{x}) = f(\mathbf{x}).$$

Show that

$$\nabla \cdot (\varphi(\mathbf{x})\nabla G(\mathbf{x};\mathbf{x}_0) - G(\mathbf{x};\mathbf{x}_0)\nabla\varphi(\mathbf{x})) = -f(\mathbf{x})G(\mathbf{x};\mathbf{x}_0)$$

at each point $\mathbf{x}$ in Ω, except $\mathbf{x} = \mathbf{x}_0$. [You may assume the identity $\nabla \cdot (h\mathbf{F}) = \nabla h \cdot \mathbf{F} + h\nabla \cdot \mathbf{F}$ for all scalar fields h and vector fields $\mathbf{F}$.]

(d) Let S denote a (small) sphere of radius $\varepsilon > 0$, centre $\mathbf{x}_0$; denote its surface by ∂S. Let $\mathbf{n}$ denote the unit vector that is outward pointing at each point of $\partial\Omega$. Let $\Omega_1 = \Omega - S = \{\mathbf{x} \in \Omega \text{ and } \mathbf{x} \notin S\}$. Show that

$$\begin{aligned} &\int_{\partial\Omega} \left(\varphi(\mathbf{x}) \frac{\partial G(\mathbf{x};\mathbf{x}_0)}{\partial n} - G(\mathbf{x};\mathbf{x}_0) \frac{\partial \varphi(\mathbf{x})}{\partial n} \right) dS_{\mathbf{x}} \\ &- \int_{\partial S} \left(\varphi(\mathbf{x}) \frac{\partial G(\mathbf{x};\mathbf{x}_0)}{\partial r} - G(\mathbf{x};\mathbf{x}_0) \frac{\partial \varphi(\mathbf{x})}{\partial r} \right) dS_{\mathbf{x}} \\ = &- \int_{\Omega_1} f(\mathbf{x})\, G(\mathbf{x};\mathbf{x}_0)\, d\mathbf{x}, \end{aligned}$$

where $\frac{\partial \varphi}{\partial r}$ and $\frac{\partial G}{\partial r}$ denotes derivatives in the direction of the outward normal to the surface of the sphere.

(e) Show that

$$\lim_{\varepsilon \to 0} \int_{\partial S} \varphi(\mathbf{x}) \frac{\partial G(\mathbf{x};\mathbf{x}_0)}{\partial r} dS_{\mathbf{x}} = \varphi(\mathbf{x}_0),$$

and

$$\lim_{\varepsilon \to 0} \int_{\partial S} G(\mathbf{x};\mathbf{x}_0) \frac{\partial \varphi(\mathbf{x})}{\partial r} dS_{\mathbf{x}} = 0.$$

Deduce that

$$\begin{aligned} \varphi(\mathbf{x}_0) = &\int_{\partial\Omega} \left(\varphi(\mathbf{x}) \frac{\partial G(\mathbf{x};\mathbf{x}_0)}{\partial n} - G(\mathbf{x};\mathbf{x}_0) \frac{\partial \varphi(\mathbf{x})}{\partial n} \right) dS_{\mathbf{x}} \\ &+ \int_{\Omega} f(\mathbf{x})\, G(\mathbf{x};\mathbf{x}_0)\, d\mathbf{x}. \end{aligned}$$

[You may assume that

$$\int_S f(\mathbf{x})\, G(\mathbf{x};\mathbf{x}_0)\, d\mathbf{x} \to 0 \text{ as } \varepsilon \to 0.]$$