# FROM TAMSAMANI'S WEAK 2-CATEGORIES TO BICATEGORIES

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Abstract. We give a conceptual description of the bicategory associated to a Tamsamani's weak 2-category, and we associate functorially to a morphism of Tamsamani's weak 2-categories a homomorphism of the corresponding bicategories.

## 1. Introduction.

The goal of this paper is to construct a functor from the category $\mathcal{N}_2$ of Tamsamani's 2-nerves and their morphisms to the category Bicat of bicategories and homomorphisms. On objects, this functor associates to a Tamsamani's 2-nerve a bicategory as constructed by Tamsamani in [5] and [6]. Our approach is however more conceptual, and allows to associate to a morphism of 2-nerves a homomorphism of the corresponding bicategories. The last point is missing in [5],[6].

We proceed in two steps. First we construct a functor from Tamsamani's weak 2-categories to a certain subcategory $S(\mathcal{N}_2)$ of pseudo-functors from $\Delta^{op}$ to $\mathrm{Cat}$. This is done in Section 3, using the categorical property of "transport of structure along an adjunction" (see Section 2). In Section 4, we construct a functor from $S(\mathcal{N}_2)$ to the category of bicategories and homomorphisms.

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## 2. Preliminaries.

We recall the definition of Tamsamani weak 2-category or, as they are called in [5], [6], 2-nerves. A *2-nerve* is a simplicial object in Cat $\phi : \Delta^{op} \to \mathrm{Cat}$ such that

i) $\phi_0$ is a discrete category.
ii) For each $n \geq 2$ the Segal maps $\eta_n : \phi_n \to \phi_1 \times_{\phi_0} \overset{n}{\cdots} \times_{\phi_0} \phi_1$ are equivalences of categories.

Given a 2-nerve $\phi : \Delta^{op} \to \mathrm{Cat}$, since $\phi_0$ is a discrete category, we have

$$\phi_1 = \coprod_{x,y \in \phi_0} \phi_{(x,y)}$$

where $\phi_{(x,y)}$ is the full subcategory of $\phi_1$ whose objects are $z \in Ob\,\phi_1$, $\partial_0 z = x$, $\partial_1 z = y$, $\partial_0, \partial_1 : \phi_1 \to \phi_0$ face operators.

Let $T\phi_n$ be the set of isomorphisms classes of objects of the category $\phi_n$. It is easy to see that the simplicial set $T\phi : \Delta^{op} \to \mathbf{Set}$, $T\phi([n]) = T\phi_n$ is the nerve of a category. A morphism of 2-nerves $f : \phi \to \phi'$ is called an *external equivalence* if

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for all $x, y \in \phi_0$ the functors $f(x,y) : \phi_{(x,y)} \to \phi'_{(fx,fy)}$ and $Tf : T\phi \to T'\phi$ are equivalences of categories.

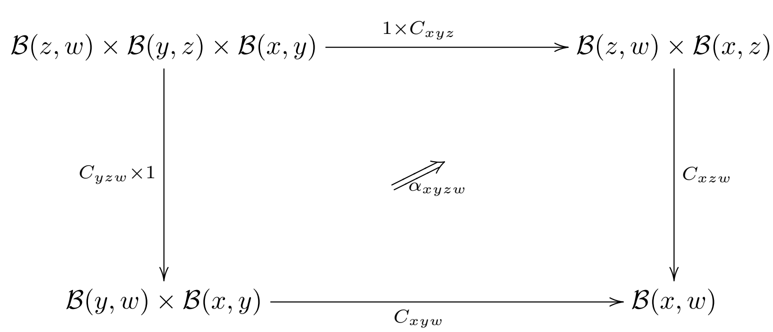
Given a 2-category $\mathcal{C}$, we denote by $Ps[\mathcal{C}, \mathrm{Cat}\,]$ the 2-category whose objects are the pseudo-functors from $\mathcal{C}$ to Cat , morphisms the pseudo-natural transformations and 2-cells the modifications. We denote by $[\mathcal{C}, \mathrm{Cat}\,]$ the 2-category of 2-functors, 2-natural transformations and modifications. We refer the reader to [1] and [3] for the definitions of these notions. It can be shown that there is a 2-category $Ps[\mathcal{C}, \mathrm{Cat}\,]$ whose 0-cells are the pseudo-functors, 1-cells are pseudo-natural transformations and 2-cells are the modifications.

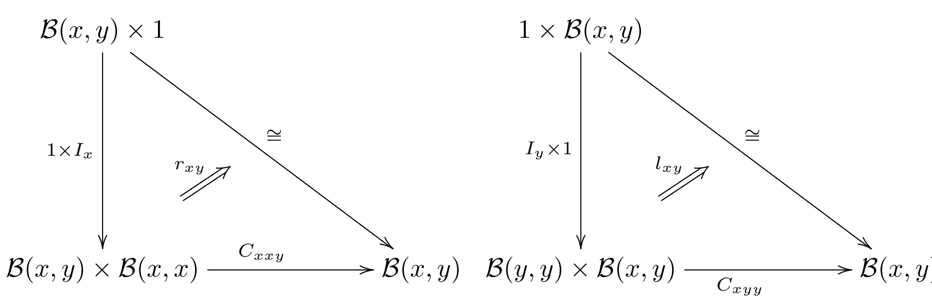
A *bicategory* $\mathcal{B}$ consists of the following data:

- a set of objects, or 0-cells, $Ob\,\mathcal{B}$;
- for each $x, y \in Ob\,\mathcal{B}$, categories $\mathcal{B}(x,y)$, whose objects are called 1-cells and morphisms 2-cells;
- for each $x, y, z \in Ob\,\mathcal{B}$, functors

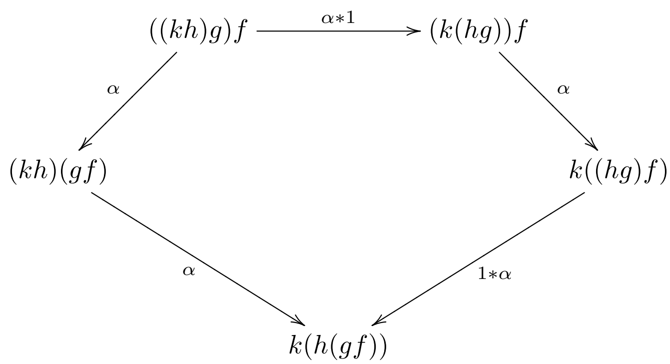
$$\begin{aligned} C_{xyz} : \mathcal{B}(y,z) \times \mathcal{B}(x,y) &\to \mathcal{B}(x,z) \\ (g,f) &\to gf \\ (\beta,\alpha) &\to \beta * \alpha \\ I_x : 1 &\to \mathcal{B}(x,x); \end{aligned}$$

- for each $x, y, z, w \in Ob\,\mathcal{B}$, natural isomorphisms

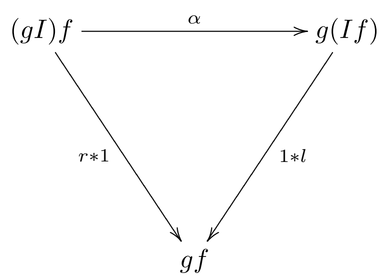


These data have to satisfy coherence axioms, given by the commutativity of the following diagrams




(1)



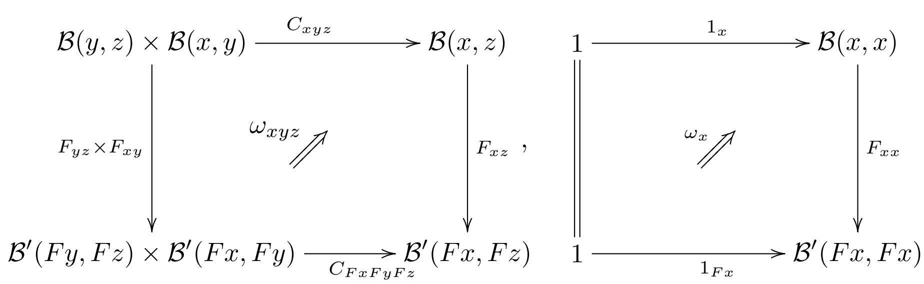

(2)

A *homomorphism of bicategories* from $\mathcal{B}$ to $\mathcal{B}'$ consists of the following data:

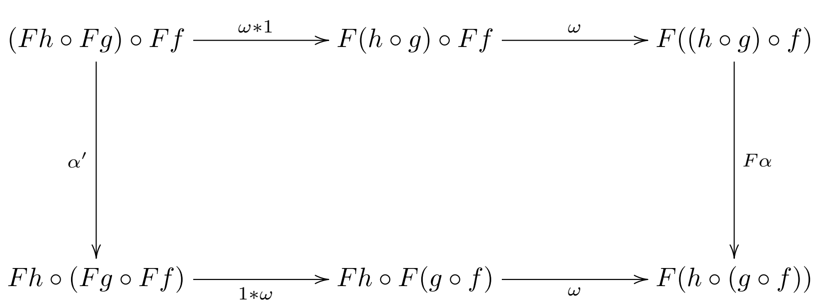
- a function $F : Ob\,\mathcal{B} \to Ob\,\mathcal{B}'$;
- for each $x, y \in Ob\,\mathcal{B}$, functors

$$F_{xy} : \mathcal{B}(x,y) \to \mathcal{B}'(Fx, Fy);$$

- natural isomorphisms




These data have to satisfy the condition that the following diagrams commute

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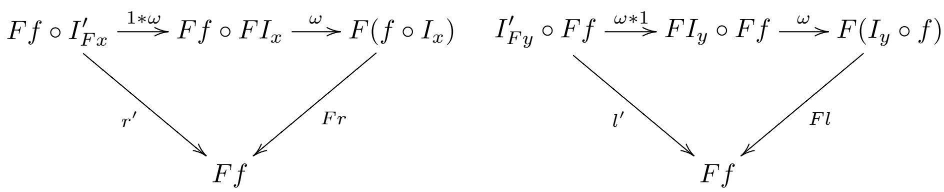


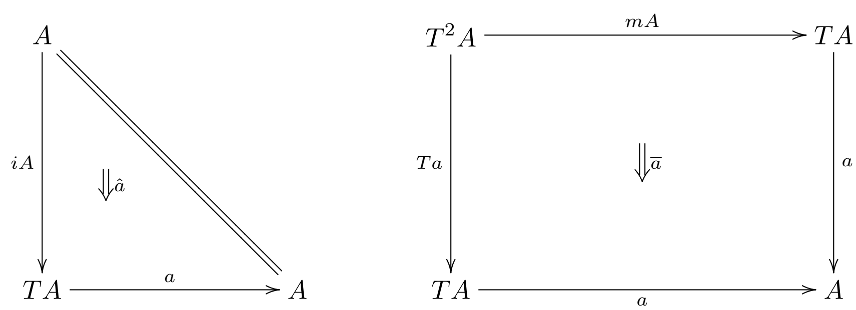

(4)

We denote by Bicat the category of bicategories and homomorphisms. An arrow $b : x \to y$ in a bicategory is an *equivalence* if there exists an arrow $b' : y \to x$ and invertible 2-cells $bb' \cong \mathrm{id}_y$ and $b'b \cong \mathrm{id}_x$. A homomorphism of bicategories $F : \mathcal{B} \to \mathcal{B}'$ is said to be a *biequivalence* if for all $x, y \in Ob\,\mathcal{C}$ the functors $\mathcal{B}(x, y) \to \mathcal{B}'(Fx, Fy)$ are equivalences of categories and if, for all $z \in Ob\,\mathcal{B}'$ there exists $x \in Ob\,\mathcal{B}$ and an equivalence $Fx \to z$.

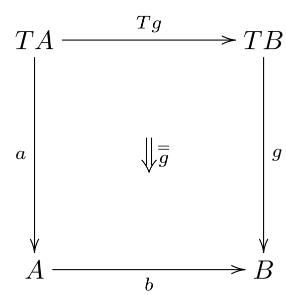
We will use the notion of 2-monad, their algebras and pseudo-algebras. Recall that a *2-monad* on a 2-category $\mathcal{A}$ is a 2-functor $T : \mathcal{A} \to \mathcal{A}$ with 2-natural transformations $m : T^2 \to T$ and $i : 1 \to T$ satisfying the equations

$$m \cdot Ti = 1, \qquad m \cdot iT = 1, \qquad m \cdot Tm = m \cdot mT.$$

Given a 2-monad $T$ on $\mathcal{A}$ a *T-algebra* is defined as in the case of ordinary monads. A *pseudo-T-algebra* is given by $(A, a, \hat{a}, \overline{a})$ where $A \in \mathcal{A}$, $a : TA \to A$, $\hat{a}$, $\overline{a}$ are invertible 2-cells




satisfying certain coherence axioms [2]. A *morphism of pseudo-T-algebras* from $(A, a, \hat{a}, \overline{a})$ to $(B, b, \hat{b}, \overline{b})$ is given by $(g, \overline{\overline{g}})$ where $g : A \to B$ and $\overline{\overline{g}}$ is an invertible 2-cell



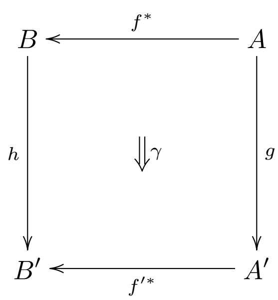

satisfying certain coherence axioms [2]. In Section 3 we will use a categorical property known as "transport of structure along an adjunction" given by the following theorem:

**Theorem 2.1.** [2, Theorem 6.1] *Given an equivalence* $\eta,\ \varepsilon : f, f^* : A \to B$ *in the complete and locally small 2-category* $\mathcal{A}$*, and an algebra* $(A, a)$ *for the monad* $T = (T, i, m)$ *on* $\mathcal{A}$*, the equivalence enriches to an equivalence*

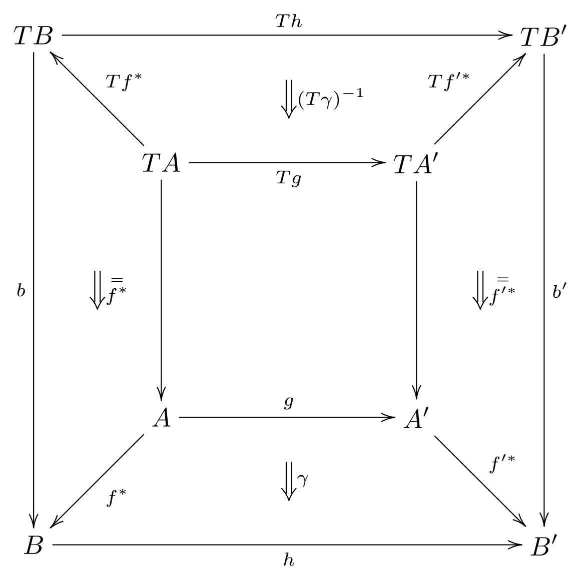
$$\eta, \varepsilon : (f, \overline{\overline{f}}) \vdash (f^*, \overline{\overline{f^*}}) : (A, a) \to (B, b, \hat{b}, \overline{b})$$

*in Ps-T-Alg, where* $\hat{b} = \eta$, $\overline{b} = f^* a \cdot T\varepsilon \cdot Ta \cdot T^2 f$, $\overline{\overline{f}} = \varepsilon^{-1} a \cdot Tf$, $\overline{\overline{f^*}} = F^* a \cdot T\varepsilon$.

Let now $\eta', \varepsilon' : f' \vdash f'^* : A' \to B'$ be another equivalence in $\mathcal{A}$ and let $(B', b', \hat{b}', \overline{b'})$ be the corresponding pseudo-$T$-algebra as in Theorem 2.1. Suppose $g : (A, a) \to (A', a')$ is a morphism in $\mathcal{A}$ and $\gamma$ is an invertible 2-cell in $\mathcal{A}$




Let $\overline{\gamma}$ be the invertible 2-cell given by the following pasting:




Then it is not difficult to show that $(h, \overline{\gamma}) : (B, b, \hat{b}, \overline{b}) \to (B', b', \hat{b}', \overline{b'})$ is a pseudo-$T$-algebra morphism.

## 3. From 2-nerves to pseudo-functors.

We start by discussing a general fact, which is essentially known. As sketched in the proof below, it is an instance of "transport of structure along an adjunction". Alternatively, a direct but rather tedious proof by pasting diagrams is also possible.

**Lemma 3.1.** *Let $\mathcal{C}$ be a small 2-category, $F, F' : \mathcal{C} \to \mathrm{Cat}$ be 2-functors, $\alpha : F \to F'$ a 2-natural transformation. Suppose that, for all objects $C$ of $\mathcal{C}$, the following conditions hold:*

i) *$G(C)$, $G'(C)$ are objects of Cat and there are adjoint equivalences of categories $\mu_C \vdash \eta_C$, $\mu'_C \vdash \eta'_C$,*
$$\mu_C : G(C) \rightleftarrows F(C) : \eta_C \qquad \mu'_C : G'(C) \rightleftarrows F'(C) : \eta'_C,$$
ii) *there are functors $\beta_C : G(C) \to G'(C)$,*
iii) *there is an invertible 2-cell*
$$\gamma_C : \beta_C \eta_C \Rightarrow \eta'_C \alpha_C.$$

*Then*

a) *There exists a pseudo-functor* $G : \mathcal{C} \to \mathrm{Cat}$ *given on objects by* $G(C)$*, and pseudo-natural transformations* $\eta : F \to G$*,* $\mu : G \to F$ *with* $\eta(C) = \eta_C$*,* $\mu(C) = \mu_C$*; these are part of an adjoint equivalence* $\mu \vdash \eta$ *in the 2-category* $Ps[\mathcal{C}, \mathrm{Cat}]$.

b) *There is a pseudo-natural transformation* $\beta : G \to G'$ *with* $\beta(C) = \beta_C$ *and an invertible 2-cell in* $Ps[\mathcal{C}, \mathrm{Cat}]$*,* $\gamma : \beta\eta \Rightarrow \eta\alpha$ *with* $\gamma(C) = \gamma_C$.

**Proof.** Recall [4] that the functor 2-category $[\mathcal{C}, \mathrm{Cat}]$ is 2-monadic over $[|\mathcal{C}|, \mathrm{Cat}]$, where $|C|$ is the set of objects in $\mathcal{C}$. Let $T$ be the 2-monad; then the pseudo-$T$-algebras are precisely the pseudo-functors from $\mathcal{C}$ to Cat. Let

$$\mathcal{U} : Ps\text{-}T\text{-}Alg \equiv Ps[\mathcal{C}, \mathrm{Cat}] \to [|\mathcal{C}|, \mathrm{Cat}]$$

be the forgetful functor.

Then the adjoint equivalences $\mu_C \vdash \eta_C$ amount precisely to an adjoint equivalence in $[|\mathcal{C}|, \mathrm{Cat}]$, $\mu_0 \vdash \eta_0$, $\mu_0 : G_0 \rightleftarrows \mathcal{U}F : \eta_0$ where $G_0(C) = G(C)$ for all $C \in |\mathcal{C}|$. By Theorem 2.1, this equivalence enriches to an adjoint equivalence $\mu \vdash \eta$ in $Ps[\mathcal{C}, \mathrm{Cat}]$

$$\mu : G \rightleftarrows F : \eta$$

between $F$ and a pseudo-functor $G$; it is $\mathcal{U}G = G_0$, $\mathcal{U}\eta = \eta_0$, $\mathcal{U}\mu = \mu_0$; hence on objects $G$ is given by $G(C)$, and $\eta(C) = \mathcal{U}\eta(C) = \eta_C$, $\mu(C) = \mathcal{U}\mu(C) = \mu_C$.

Let $\nu_C : \mathrm{id}_{G(C)} \Rightarrow \eta_C\mu_C$ and $\varepsilon_C : \mu_C\eta_C \Rightarrow \mathrm{id}_{F(C)}$ be the unit and counit of the adjunction $\mu_C \vdash \eta_C$. From Theorem 2.1, given a morphism $f : C \to D$ in $\mathcal{C}$, it is

$$G(f) = \eta_D F(f) \mu_C$$

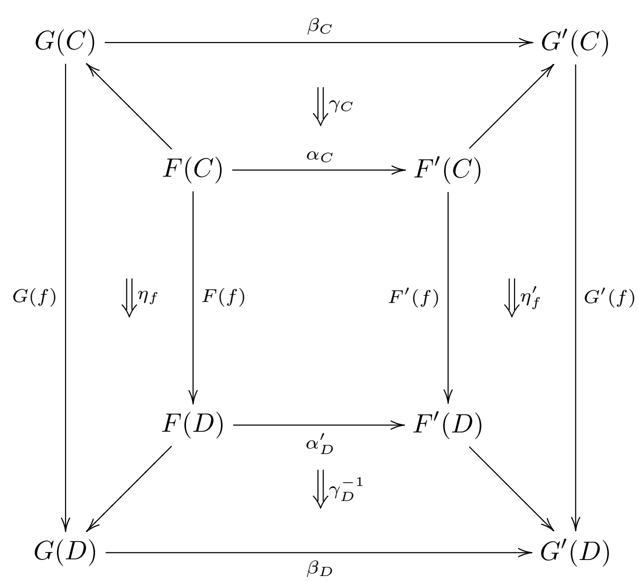
and we have natural isomorphisms:

$$\eta_f : G(f)\eta_C = \eta_D F(f)\mu_C\eta_C \xRightarrow{\eta_D F(f)\varepsilon_C} \eta_D F(f)$$
$$\mu_f : F(f)\mu_C \xRightarrow{\nu_{F(f)}\mu_C} \mu_D\eta_D F(f)\mu_C = \mu_D G(f).$$

Also, the natural isomorphism

$$\beta_f : G'(f)\beta_C \Rightarrow \beta_D G(f)$$

is the result of the following pasting




□

We now apply Lemma 3.1 to the case where $\mathcal{C} = \Delta^{op}$, considered as a 2-category with identity 2-cells; let $\phi : \Delta^{op} \to \mathrm{Cat}$ be a 2-nerve. By definition, for each $n \geq 2$ there is an equivalence of categories $\phi_n \simeq \phi_1 \times_{\phi_0} \overset{n}{\cdots} \times_{\phi_0} \phi_1$. We can always choose this equivalence to be an adjoint equivalence; thus let $\eta_n : \phi_n \to \phi_1 \times_{\phi_0} \overset{n}{\cdots} \times_{\phi_0} \phi_1$ be the Segal map and $\mu_n$ its left adjoint. By Lemma 3.1, we deduce that there is a pseudo-functor $\tilde{\phi} \in Ps[\Delta^{op}, \mathrm{Cat}\,]$ with

$$\tilde{\phi}_n = \begin{cases} \phi_1 \times_{\phi_0} \overset{n}{\cdots} \times_{\phi_0} \phi_1 & n \geq 2, \\ \phi_1 & n = 1, \\ \phi_0 & n = 0. \end{cases}$$

Suppose $F : \phi \to \phi'$ is a morphism of 2-nerves and let $\beta_n : \tilde{\phi}_n \to \tilde{\phi}'_n$ be

$$\beta_n = \begin{cases} (F_1, \ldots, F_1) & n \geq 2, \\ F_1 & n = 1, \\ F_0 & n = 0. \end{cases}$$

It is immediate to check from the definition of Segal map that the following diagram commutes for all $n \geq 0$

$$\begin{array}{ccc} \phi_n & \xrightarrow{F_n} & \phi'_n \\ {\scriptstyle \eta_n}\downarrow & & \downarrow{\scriptstyle \eta'_n} \\ \tilde{\phi}_n & \xrightarrow[\beta_n]{} & \tilde{\phi}'_n \end{array}$$

Thus the condition in the hypothesis of Lemma 3.1 is satisfied, with $\gamma_n$ the identity 2-cell. It follows from Lemma 3.1 that there is a pseudo-natural transformation $\beta : \tilde{\phi} \to \tilde{\phi}'$ with $\beta(F)_n = \beta_n$.

Suppose that $\phi \xrightarrow{F} \phi' \xrightarrow{F'} \phi''$ is a pair of composable morphisms of 2-nerves, then, for each $n \geq 2$

$$\beta(F'F)_n = ((F'F)_1, \ldots, (F'F)_1) = (F'_1, \ldots, F'_1)(F_1, \ldots, F_1) = \beta(F')_n \beta(F)_n.$$

Therefore $\beta(F'F) = \beta(F')\beta(F)$.

Let $\mathcal{P}$ be the category whose objects are pseudo-functors $\psi : \Delta^{op} \to \mathrm{Cat}$ such that $\psi_0$ is discrete and, for each $n \geq 2$, $\psi_n = \psi_1 \times_{\psi_0} \overset{n}{\cdots} \times_{\psi_0} \psi_1$, and whose morphisms are the pseudo-natural transformations $F$ with $F_n = (F_1, \cdots, F_1)$ for $n \geq 2$.

In conclusion, we have proved the following:

**Proposition 3.2.** *There is a functor*

$$S : \mathcal{N}_2 \to \mathcal{P}.$$

*On object, $S$ associates to a 2-nerve $\phi$ a pseudo-functor $\tilde{\phi} \in Ps[\Delta^{op}, \mathrm{Cat}\,]$ with*

$$\tilde{\phi}_n = \begin{cases} \phi_1 \times_{\phi_0} \overset{n}{\cdots} \times_{\phi_0} \phi_1 & n \geq 2, \\ \phi_1 & n = 1, \\ \phi_0 & n = 0. \end{cases}$$

*To a morphism of 2-nerves $F : \phi \to \phi'$, $S$ associates a pseudo-natural transformation $\beta(F) : \tilde{\phi} \to \tilde{\phi}'$ with*

$$\beta(F)_n = \begin{cases} (F_1, \ldots, F_1) & n \geq 2, \\ F_1 & n = 1, \\ F_0 & n = 0. \end{cases}$$

## 4. From pseudo-functors to bicategories.

The aim of this section is to construct a functor from the subcategory $S(\mathcal{N}_2)$ of $\mathcal{P}$ to the category Bicat of bicategories and homomorphisms.

Let $\tilde{\phi}:\Delta^{op}\rightarrow \mathrm{Cat}$ be the object of $S(\mathcal{N}_2)$ corresponding to $\phi\in\mathcal{N}_2$. Since $\phi_0$ is discrete, we have

$$\phi_1=\coprod_{x,y\in\phi_0}\phi_{(x,y)},$$

where $\phi_{(x,y)}$ is the full subcategory of $\phi_1$, whose objects are $\{z\in Ob\,\phi_1 \mid \partial_0 z=x,\ \ \partial_1 z=y\}$, $\partial_0,\partial_1$ face maps.

We construct a bicategory $R\tilde{\phi}$ as follows. The set of 0-cells of $R\tilde{\phi}$ is $\phi_0$. Given $x,y\in\phi_0$, the 1-cells between $x$ and $y$ are the objects of $\phi_{(x,y)}$, the 2-cells are the morphisms of $\phi_{(x,y)}$. Vertical composition of 2-cells is composition of morphisms in $\phi_{(x,y)}$. Horizontal composition of 1 and 2-cells is given by the map $c=\tilde{\phi}(\delta_{02}):\tilde{\phi}_2=\phi_1{\times_{\phi_0}}\phi_1\rightarrow\phi_1$ induced by $\delta_{02}:[1]\rightarrow[2],\ \ 0\rightarrow 0,\ \ 1\rightarrow 2$. Consider the maps

$$\tilde{\phi}_3=\phi_1{\times_{\phi_0}}\phi_1{\times_{\phi_0}}\phi_1 \underset{\mathrm{id}\times c}{\overset{c\times\mathrm{id}}{\rightrightarrows}} \phi_1{\times_{\phi_0}}\phi_1=\tilde{\phi}_2\ .$$

It is easily checked that there are natural isomorphisms $c\times\mathrm{id}\Rightarrow\tilde{\phi}(\delta_{013}),\ \mathrm{id}\times c\Rightarrow\tilde{\phi}(\delta_{023})$, where $\delta_{ijk}:[2]\rightarrow[3]$ is the map $0\rightarrow i,\ \ 1\rightarrow j,\ \ 2\rightarrow k$. Let $d:\tilde{\phi}_3\rightarrow\phi_1$ be $d=\tilde{\phi}(\delta_{03}),\ \ \delta_{03}:[1]\rightarrow[3],\ \ 0\rightarrow 0,\ \ 1\rightarrow 3$. Since $\delta_{03}=\delta_{013}\delta_{02}=\delta_{023}\delta_{02}$ and $\tilde{\phi}$ is a pseudo-functor, we have natural isomorphisms given by the composites:

$$\begin{aligned}\alpha_1 &: d\Rightarrow c\ \tilde{\phi}(\delta_{013})\Rightarrow c(c\times\mathrm{id})\\ \alpha_2 &: d\Rightarrow c\ \tilde{\phi}(\delta_{023})\Rightarrow c(\mathrm{id}\times c)\end{aligned}$$

and therefore a natural isomorphism $\alpha=\alpha_2\alpha_1^{-1}$:

$$\begin{array}{ccc}\phi_1{\times_{\phi_0}}\phi_1{\times_{\phi_0}}\phi_1 & \xrightarrow{\ \mathrm{id}\times c\ } & \phi_1{\times_{\phi_0}}\phi_1\\ {\scriptstyle c\times\mathrm{id}}\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle c}\\ \phi_1{\times_{\phi_0}}\phi_1 & \xrightarrow[\ c\ ]{} & \phi_1\end{array}$$

By restriction, this gives the associativity isomorphism:

$$\begin{array}{ccc}\phi_{(z,w)}\times\phi_{(y,z)}\times\phi_{(x,y)} & \xrightarrow{\ \mathrm{id}\times c\ } & \phi_{(z,w)}\times\phi_{(x,z)}\\ {\scriptstyle c\times\mathrm{id}}\downarrow & \overset{\alpha_{xyzw}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle c}\\ \phi_{(y,w)}\times\phi_{(x,y)} & \xrightarrow[\ c\ ]{} & \phi_{(x,w)}\end{array} \tag{5}$$

Let $i_0=\tilde{\phi}(\sigma_0),\ \ \ i_1=\tilde{\phi}(\sigma_1):\phi_1\rightarrow\phi_1{\times_{\phi_0}}\phi_1$ where $\sigma_0,\sigma_1:[2]\rightarrow[1]$ are the maps $0\rightarrow 1,\ \ 1\rightarrow 1,\ \ 2\rightarrow 1$ and $0\rightarrow 0,\ \ 1\rightarrow 0,\ \ 2\rightarrow 1$ respectively. Since $\sigma_0\delta_{02}=\sigma_1\delta_{02}=\mathrm{id}$ and $\tilde{\phi}$ is a pseudo-functor, we have natural isomorphisms

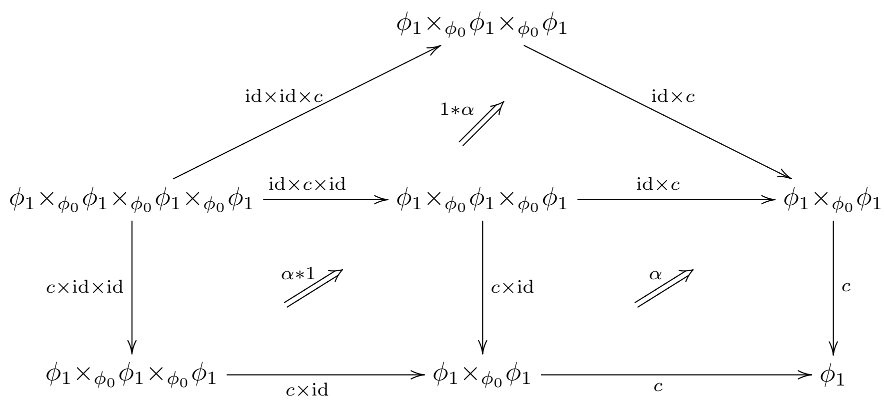
$$i_0 c\Rightarrow\mathrm{id},\qquad i_1 c\Rightarrow\mathrm{id}.$$

By restriction, we obtain identity isomorphisms:


(6)

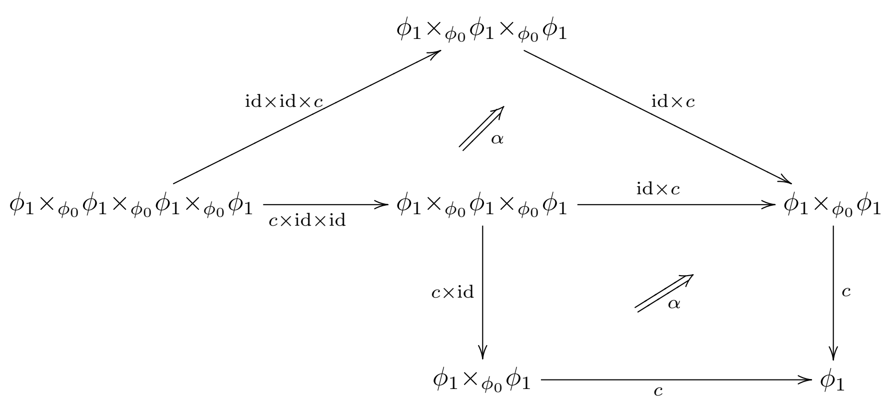
We need to show that the associativity and unit isomorphisms (5) and (6) satisfy the coherence axioms; that is, we need to show that the diagrams (1) and (2) commute.

In (1) the composite $(1*\alpha)\alpha(\alpha*1)$ is obtained by pasting the following diagram:



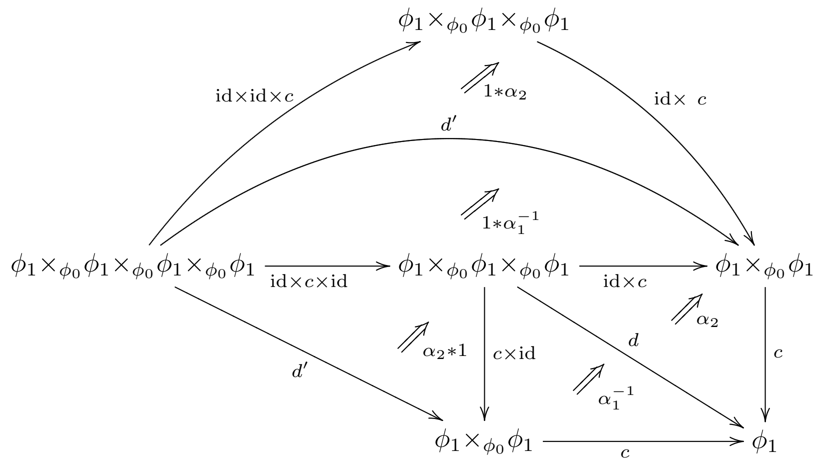

(7)

The composite $\alpha\alpha$ in (1) is the result of pasting the following diagram:

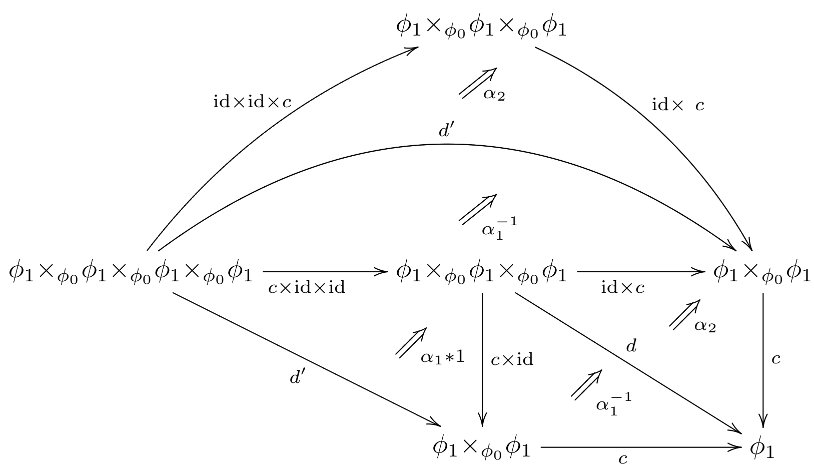



(8)

Let $d' = (d \times \mathrm{id}) : \phi_1 \times_{\phi_0} \phi_1 \times_{\phi_0} \phi_1 \times_{\phi_0} \phi_1 \to \phi_1 \times_{\phi_0} \phi_1$. Because of the way in which the associativity isomorphism are constructed, showing that the pasting of (7) and of (8) gives the same result is equivalent to showing that the pasting of the following diagrams coincides.

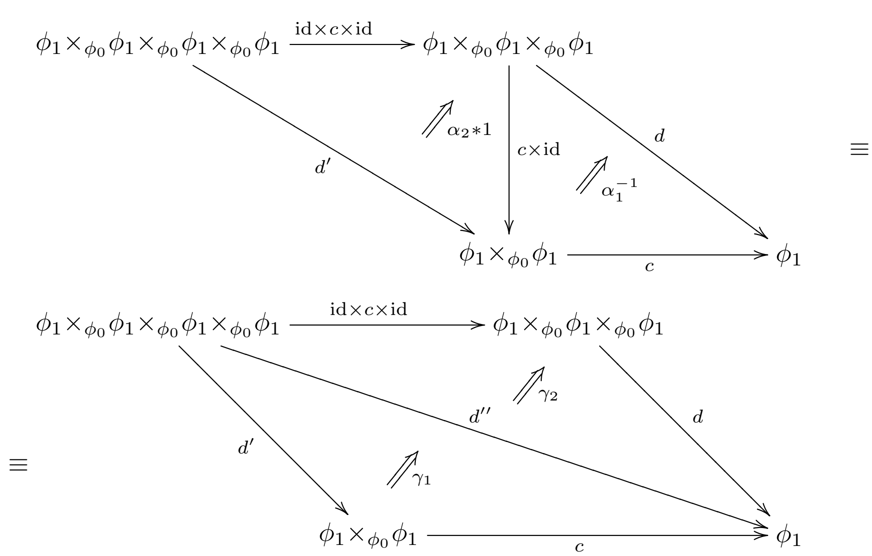



(9)

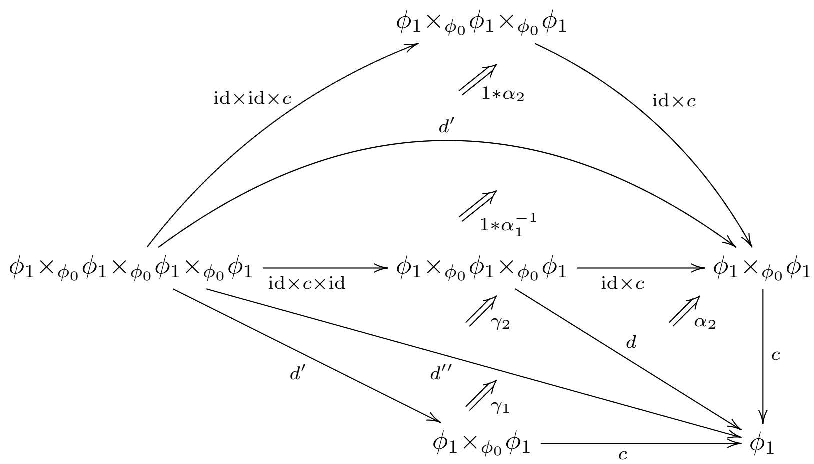



(10)

Let $d''=\tilde{\phi}(\delta_{04}):\phi_1\times_{\phi_0}\phi_1\times_{\phi_0}\phi_1\times_{\phi_0}\phi_1\rightarrow\phi_1,\ \ \delta_{04}:[1]\rightarrow[4],\, 0\rightarrow 0,\ \ 1\rightarrow 4$. Since $\tilde{\phi}$ is a pseudo-functor we have the following identity:

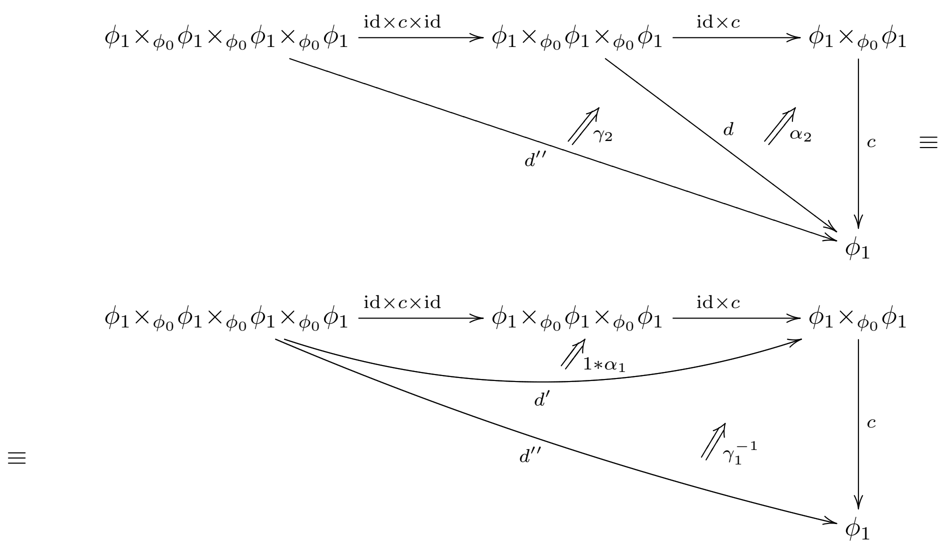


hence the pasting diagram (9) becomes

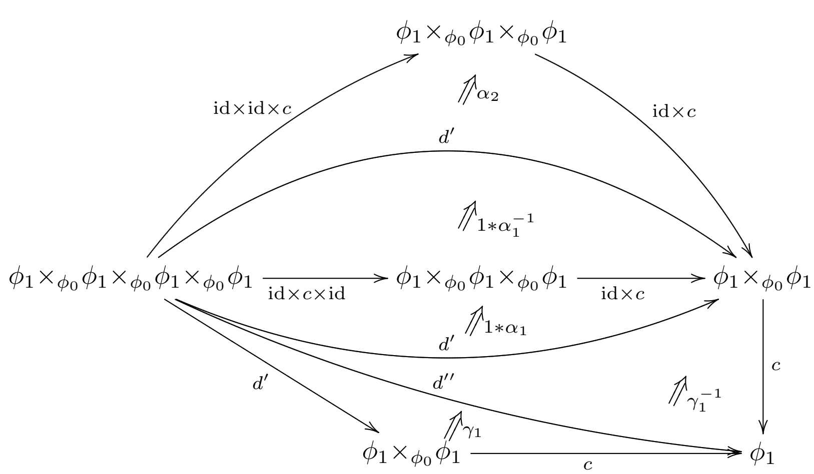



(11)

Since $\tilde{\phi}$ is a pseudo-functor, we also have the identity

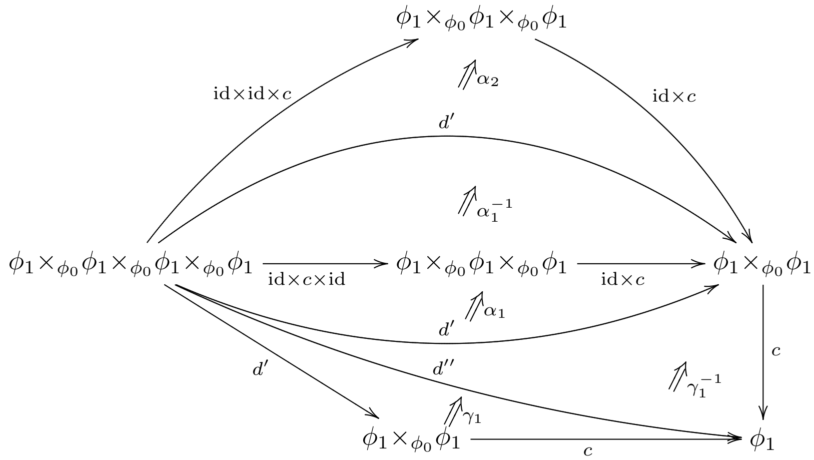



Hence (11) is equivalent to:




(12)

A similar argument shows that (10) is equivalent to the following diagram:



(13)

It is then clear that the result of the pasting of (12) and of (13) coincides.

This proves the commutativity of (1). The proof of the commutativity of (2) is similar. Hence $R\tilde{\phi}$ is a bicategory.

Let $\tilde{\phi}, \tilde{\phi}'$ be objects of $S(\mathcal{N}_2)$ and let $F : \tilde{\phi} \Rightarrow \tilde{\phi}'$ be a morphism in $S(\mathcal{N}_2)$. We are going to show that $F$ induces a homomorphism on bicategories

$$R(F) : R(\tilde{\phi}) \to R(\tilde{\phi}').$$

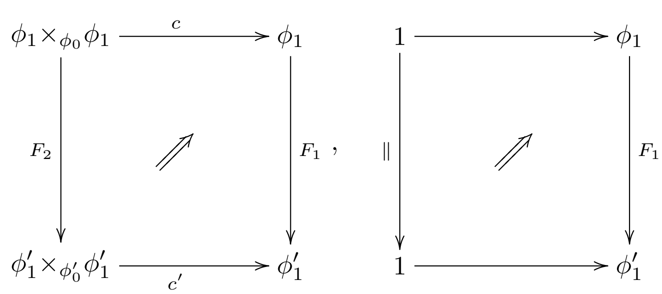
On objects, we have the function

$$R(F)_0 = F_0 : \phi_0 \to \phi_0'.$$

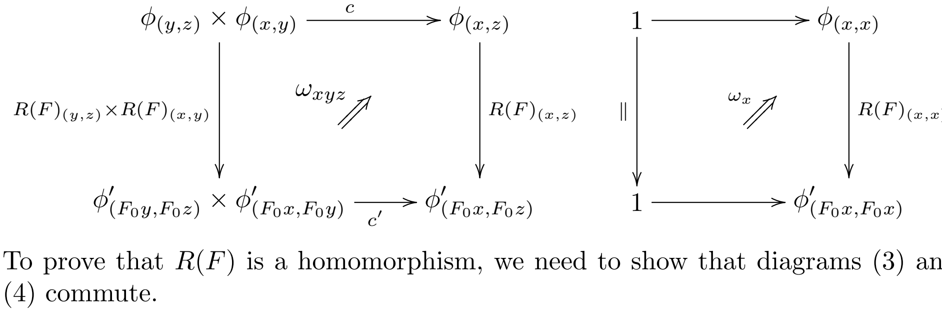
Since $\phi_0, \phi_0'$ are discrete, we have $F_0 \partial_i = \partial_i' F_1$ where $\partial_i, \partial_i' : \phi_1 \to \phi_0$ are induced by $\delta_i : [0] \to [1], \quad 0 \to i, \quad i = 0, 1$. Therefore for each $x, y \in \phi_0$ the functor $F_1 : \phi_1 \to \phi_1'$ induces by restriction a functor

$$R(F)_{(x,y)} : \phi_{(x,y)} \to \phi'_{(F_0 x, F_0 y)}.$$

Since $F$ is a pseudo-natural transformation, there are natural isomorphisms

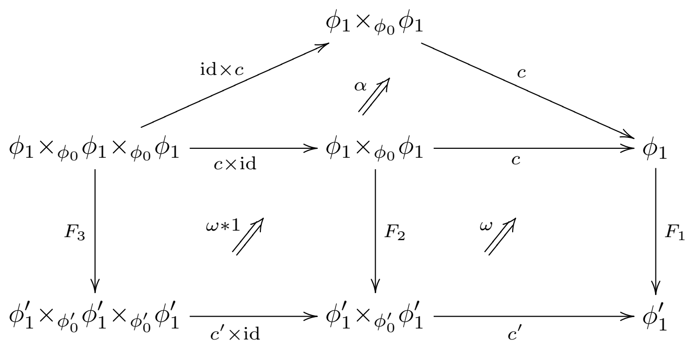



By restriction, these give natural isomorphisms, for each $x, y, z \in \phi_0$:



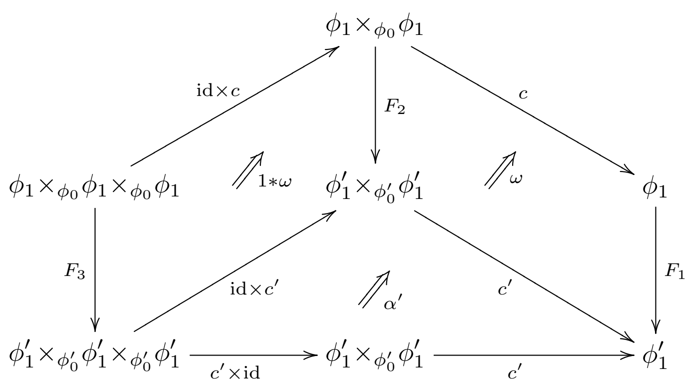

To prove that $R(F)$ is a homomorphism, we need to show that diagrams (3) and (4) commute.

The composite $R(F)\alpha\omega(\omega{*}1)$ in (3) is the result of pasting the following diagram:



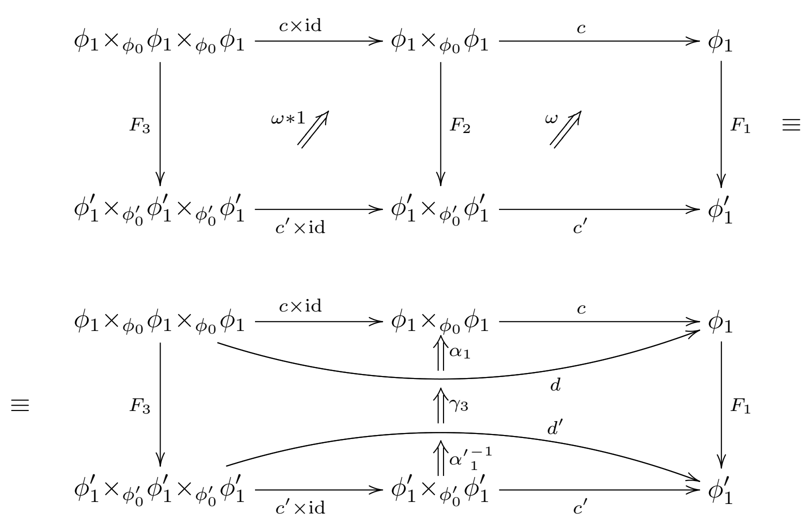
(14)

The composite $\omega(1 * \omega)\alpha'$ in (4) is given by pasting of the following diagram:



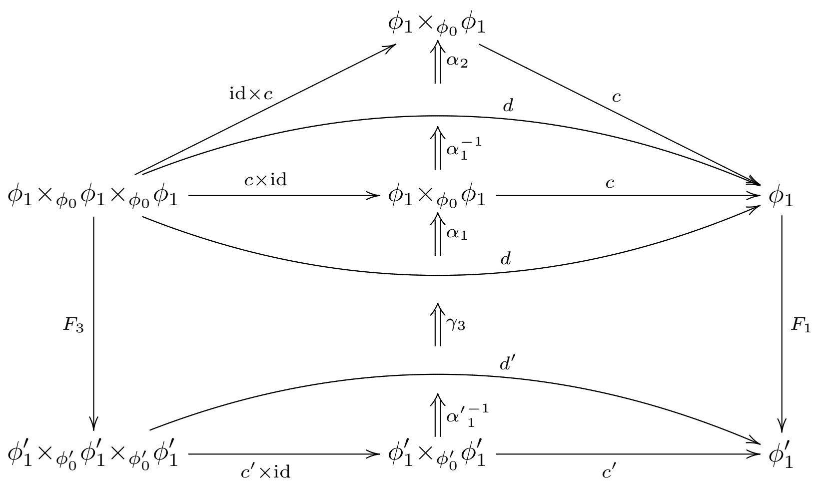
(15)

Since $F$ is a pseudo-natural transformation, we have the following identity, where the maps $d : \tilde{\phi}_3 = \phi_1{\times}_{\phi_0}\phi_1{\times}_{\phi_0}\phi_1 \to \phi_1$ and $d' : \tilde{\phi}'_3 \to \phi'_1$ are induced by $\delta_{03} = [1] \to [3], \;\; 0 \to 0, \;\; 1 \to 3$.



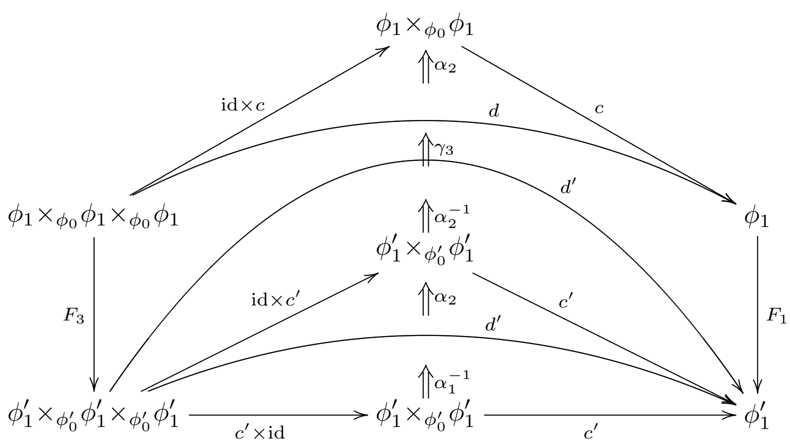
A similar identity holds for the top part of the pasting diagram (15).

From the definition of the associativity isomorphism, the diagram (14) is therefore equivalent to the following:




(16)

Similarly, the diagram (15) is equivalent to the following:




(17)

We see that result of pasting of (16) and of (17) coincides.

This proves the commutativity of (3). The commutativity of (4) is proved similarly. Hence $R(F)$ is a bicategory homomorphism. In conclusion we have built a functor $R : S(\mathcal{N}_2) \to \text{Bicat}$.

From the results of Sections 3 and 4 we finally obtain:

**Theorem 4.1.** *There is a functor*

$$Bic : \mathcal{N}_2 \to \text{Bicat}$$

*from the category of Tamsamani's 2-nerves to the category of bicategories and homomorphisms. On objects, $Bic$ associates to a 2-nerve a bicategory as constructed in* [5]*. Further, $Bic$ sends external equivalences to biequivalences.*

**Proof.** Let $Bic = R \circ S$. By construction, the bicategory $Bic\,\phi$ coincides with the one constructed in [5] from the 2-nerve $\phi$.

Suppose $F : \phi \to \phi'$ is an external equivalence in $\mathcal{N}_2$. By definition, this means that

i) for each $x, y \in \phi_0$, $\phi_{(x,y)} \to \phi'_{(F_0x, F_0y)}$ is an equivalence of categories.
ii) $TF : T\phi \to T\phi'$ is an equivalence of categories.

From the definition of $Bic\,\phi$, i) means that $(Bic\,\phi)_{(x,y)} \to (Bic\,\phi')_{(F_0x,F_0y)}$ is an equivalence of categories; from ii) we have that, for each $z \in \phi'_0$, there is $x \in \phi_0$ and an isomorphism $F_0x \cong z$ in the category $T\phi'$; this means there are arrows $b : F_0x \to z$, $b' : z \to F_0x$ with $[b'][b] = \mathrm{id} = [b][b']$ where $[b]$ (resp. $[b']$) is the isomorphism class of objects in the category $\phi_1$, represented by $b \in \phi_{10}$ (resp. $b' \in \phi'_{10}$). Hence there are invertible 2-cells $b'b \cong \mathrm{id}_{F_0x}$, $bb' \cong \mathrm{id}_z$. Thus $F_0x$ and $z$ are equivalent in the bicategory $Bic\,\phi$. By definition, $Bic\,F$ is therefore a biequivalence. □

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