Math 338 - Algebra IIIB 2004d2

Assignment M1

Due 6pm Wednesday 18 August 2004.

1. If $\alpha^3 = 2\alpha + 3$ then write α^4, α^5, α^6, and α^{-1} in the form $a\alpha^2 + b\alpha + c$ with a, b, and c rational.

2. Show that $\sqrt{3}$ is not in $\mathbf{Q}(\sqrt{2})$.

3. What are the quotient and remainder when you divide t^4 by $t^2 - t - 1$ in $\mathbf{F}_5[t]$?

4. Find polynomials $a(t)$ and $b(t)$ in $\mathbf{Q}[t]$ such that $t^3 a(t) - (t^2 - t - 2)b(t) = 1$.

5. Factor $t^4 - 28t - 45$ into irreducibles over $\mathbf{Q}$.

Assignment M1 Solutions

1. If $\alpha^3 = 2\alpha + 3$ then write α^4, α^5, α^6, and α^{-1} in the form $a\alpha^2 + b\alpha + c$ with a, b, and c rational.

Solution. $\alpha^4 = \alpha\alpha^3 = \alpha(2\alpha + 3) = 2\alpha^2 + 3\alpha$; $\alpha^5 = 2\alpha^3 + 3\alpha^2 = 2(2\alpha + 3) + 3\alpha^2 = 3\alpha^2 + 4\alpha + 6$; $a^6 = (a^3)^2 = 4\alpha^4 + 12\alpha + 9$. Also, from $\alpha^3 = 2\alpha + 3$ we get $\alpha(\alpha^2 - 2) = 3$, so $\alpha^{-1} = (1/3)\alpha^2 - (2/3)$.

2. Show that $\sqrt{3}$ is not in $\mathbf{Q}(\sqrt{2})$.

Solution. Suppose $\sqrt{3}$ is in $\mathbf{Q}(\sqrt{2})$, so $\sqrt{3} = a + b\sqrt{2}$ for some rational numbers a and b. Then $3 = a^2 + 2b^2 + 2ab\sqrt{2}$, so either $\sqrt{2} = (3 - a^2 - 2b^2)/2ab$, or $2ab = 0$. The first alternative is impossible, since $\sqrt{2}$ is not in $\mathbf{Q}$, while $(3 - a^2 - 2b^2)/2ab$ is. The second alternative implies $a = 0$ or $b = 0$. From $a = 0$ we get $\sqrt{3} = b\sqrt{2}$, which implies $\sqrt{6} = 2b$ is rational; from $b = 0$ we get that $\sqrt{3}$ is rational, so we just have to prove that these two numbers are irrational. More generally, let's prove that if n is an integer and $\sqrt{n}$ is not then $\sqrt{n}$ is irrational.

If n is negative then $\sqrt{n}$ is not real, hence not rational, so we may assume n is positive. We assume $\sqrt{n}$ is not an integer, so there is a positive integer r such that $r < \sqrt{n} < r + 1$. If $\sqrt{n}$ is rational then there are positive integers b such that $b\sqrt{n}$ is a positive integer; let b be the smallest such. Now let $c = b(\sqrt{n} - r)$.

Then c is positive, since $b > 0$ and $\sqrt{n} - r > 0$. Also, c is an integer, since $c = b\sqrt{n} - br$, and $b\sqrt{n}$ and br are both integers. Moreover, $c\sqrt{n}$ is an integer, since $c\sqrt{n} = bn - rb\sqrt{n}$ and bn and $b\sqrt{n}$ are both integers. Finally, $c < b$, since $\sqrt{n} - r < 1$. But then the existence of c contradicts the selection of b, so the assumption that $\sqrt{n}$ is rational is incorrect.

Another proof uses the Unique Factorization Theorem. Assume $\sqrt{n} = a/b$ for some integers a and b. Then $nb^2 = a^2$. Every prime appearing in the prime factorization of a must appear to an even power in the prime factorization of a^2; every prime appearing in the prime factorization of b must appear to an even power in the prime factorization of b^2. Thus every prime appearing in the factorization of n must appear to an even power. But then $\sqrt{n}$ is an integer.

From either proof it follows that $\sqrt{3}$ and $\sqrt{6}$ are irrational and that completes the proof that $\sqrt{3}$ is not in $\mathbf{Q}(\sqrt{2})$.

3. What are the quotient and remainder when you divide t^4 by $t^2 - t - 1$ in $\mathbf{F}_5[t]$?

Solution. $t^4 = (t^2 - t - 1)(t^2 + t + 2) + 3t + 2$.

4. Find polynomials $a(t)$ and $b(t)$ in $\mathbf{Q}[t]$ such that $t^3a(t) - (t^2 - t - 2)b(t) = 1$.

Solution. $t^3 = (t+1)(t^2 - t - 2) + (3t + 2)$; $9(t^2 - t - 2) = (3t - 5)(3t + 2) - 8$, so

$$\begin{aligned} 8 &= (3t-5)(3t+2) - 9(t^2 - t - 2) \\ &= (3t-5)(t^3 - (t+1)(t^2 - t - 2)) - 9(t^2 - t - 2) \\ &= (3t-5)(t^3) - (3t^2 - 2t + 4)(t^2 - t - 2) \end{aligned}$$

So we can take $a(t) = (3/8)t - (5/8)$ and $b(t) = (3/8)t^2 - t/4 + (1/2)$.

5. Factor $t^4 - 28t - 45$ into irreducibles over $\mathbf{Q}$.

Solution. If the polynomial has a factor of degree 1 then it has a zero, and that zero must be a factor of 45, so it must be one of the numbers ± 1, ± 3, ± 5, ± 9, ± 15, ± 45. In fact none of these numbers is a zero of the polynomial, so it has no factor of degree 1. Thus it is either irreducible or a product of two irreducibles each of degree 2.

If it is a product of two irreducible quadratics then we may assume both are monic and have integer coefficients, so $t^4 - 28t - 45 = (t^2 + at + b)(t^2 + ct + d)$, with integers a, b, c, and d. Expanding, we get the equations

$$a + c = 0 \qquad b + ac + d = 0 \qquad ad + bc = -28 \qquad bd = -45$$

Using the first equation to eliminate c we get

$$b + d = a^2 \qquad a(d - b) = -28 \qquad bd = -45$$

Using the new first equation to eliminate d we get

$$a(a^2 - 2b) = -28 \qquad b(a^2 - b) = -45$$

From the new second equation, b must be one of the numbers ± 1, ± 3, ± 5, ± 9, ± 15, ± 45. Trying them, we find that $b = -5$, $a = \pm 2$ is a solution to the second equation; $b = -5$, $a = -2$ also works in the first equation. So, $d = a^2 - b = 9$, $c = -a = 2$, and $t^4 - 28t - 45 = (t^2 - 2t - 5)(t^2 + 2t + 9)$.