Macquarie University

END-OF-YEAR EXAMINATION 2004

Course:	**MATH338 – ALGEBRA IIIB**
Date:	Monday, 17 November 2003 at 9.20 am
Time Allowed:	THREE (3) hours, plus 10 minutes reading time.
Number of questions:	SEVEN (7)
Instructions:	All questions may be attempted. The questions are of equal value. Use separate books for parts A and B. Non-programmable calculators without text retrieval are permitted.

PART A (4 Questions: 10 marks each)

1. [10 Marks]

a) Show that $R = \{ m + n\sqrt{2} \,|\, m, n \in \mathbf{Z} \}$ is a subring of the field $\mathbf{R}$ of real numbers and that an element $m + n\sqrt{2}$ is invertible in R if and only if $m^2 - 2n^2 = \pm 1$.

b) Let A be the smallest ideal of $\mathbf{Z}[x]$ that contains both 2 and x. Show that A is not a principal ideal.

2. [10 Marks]

a) Show that the ring $\mathrm{End}(\mathbf{Z}_n)$ of additive group morphisms $\mathbf{Z}_n \to \mathbf{Z}_n$ (under pointwise addition and composition as multiplication) is isomorphic to $\mathbf{Z}_n$ itself.

b) Show that $\mathrm{End}(\mathbf{Z} \oplus \mathbf{Z})$ is isomorphic to the ring $M_2(\mathbf{Z})$ of 2×2-matrices with entries in $\mathbf{Z}$.

3. [10 Marks]

Let R be a commutative ring. An element $a \in R$ is said to be *nilpotent* when there exists an integer $n > 0$ such that $a^n = 0$. Let N be the set of nilpotent elements of R.

a) Show that N is an ideal of R.

b) Show that R/N has no non-zero nilpotent elements.

c) Show that N is contained in every prime ideal P of R.

4. [10 Marks]

a) Show that, although $\mathbf{Z}$ is a subgroup of the additive group $\mathbf{Q}$ of rationals, the abelian group $\mathbf{Z}_2 \otimes_{\mathbf{Z}} \mathbf{Z}$ is not isomorphic to a subgroup of $\mathbf{Z}_2 \otimes_{\mathbf{Z}} \mathbf{Q}$.

b) Let A be an ideal of a ring R and let M be a left R-module. Prove that there is an isomorphism of left R-modules:

$$(R/A) \otimes_R M \simeq M/AM.$$

Continued on Next Page.

PART B (3 Questions: 10 marks each)

5. [10 Marks]

a) Show that these polynomials are irreducible over $\mathbf{Q}$.

(i) $t^3 - 3986t - 1$

(ii) $2t^{17} - 6t^{14} + 15t^{12} - 9t^7 + 21t^3 - 300$

b) Let α be a zero of $t^3 - t - 1$ in an extension of $\mathbf{Q}$. Express $\alpha^5 + \alpha^{-1}$ in the form $a + b\alpha + c\alpha^2$ with a, b, and c in $\mathbf{Q}$.

6. [10 Marks]

a) Suppose $[\mathbf{K} : \mathbf{F}] = 2$.

(i) Show that if α is in $\mathbf{K}$ but not in $\mathbf{F}$ then $\mathbf{K} = \mathbf{F}(\alpha)$.

(ii) Show that $\mathbf{K}$ is a normal extension of $\mathbf{F}$.

b) Show that if $\mathbf{K}$ is a finite (but not necessarily normal) extension of $\mathbf{Q}$ then there are only finitely many fields $\mathbf{E}$ with $\mathbf{Q} \subseteq \mathbf{E} \subseteq \mathbf{K}$.

7. [10 Marks]

Let $\mathbf{F}$ be a field of characteristic zero. Let d be in $\mathbf{F}$. Let p be a prime number. Assume $f(t) = t^p - d$ is irreducible over $\mathbf{F}$. Let $\mathbf{K}$ be a splitting field for f over $\mathbf{F}$.

a) Show that $\mathbf{K}$ contains a splitting field $\mathbf{E}$ for $t^p - 1$ over $\mathbf{F}$.

b) Show that the Galois group $G(\mathbf{E}/\mathbf{F})$ is abelian.

c) Find the Galois group $G(\mathbf{K}/\mathbf{E})$.

d) Prove that the Galois group $G(\mathbf{K}/\mathbf{F})$ is solvable. Do not use the theorem relating radical extensions to solvable groups. State carefully any theorems you do use, and verify that the hypotheses of these theorems are met.