MATH338 Algebra IIIB ASSIGNMENT S3 *Solutions*

1. Show that $\mathbf{Z}_3$ is not a free $\mathbf{Z}$-module.

Solution

In a free $\mathbf{Z}$-module M, if $x \in M$ is part of a basis then $n x = 0$ implies $n = 0$ (or else we would have two different ways to write 0 as a $\mathbf{Z}$-linear combination of basis elements). In $\mathbf{Z}_3$, $3 \times 1 = 0$ and $3 \times 2 = 0$, so neither 1 nor 2 can be part of a basis. So $\mathbf{Z}_3$ has no basis; so it is not free.

2. Let M be a left R-module and let I be an ideal of R with $I \subseteq \mathrm{Ann}(M)$.
   (a) Show that M has a natural structure as a left (R/I)-module.
   (b) Show that any left (R/I)-module N is naturally a left R-module.
   (c) For N as in (b), show that $\mathrm{Hom}_{R/I}(M,N) = \mathrm{Hom}_R(M,N)$.

Solution

(a) For $r+I \in R/I$ and $m \in M$, define $(r+I)\, m = r\, m$. This is well defined since, if $r' = r + a$ with $a \in I$, then $r'm = (r+a)m = rm + am = rm$ since $a \in \mathrm{Ann}(M)$. The action properties follow since they hold for R acting on M.

(b) If $f : R \longrightarrow S$ is a ring morphism, every S-module N becomes an R-module via $rn = f(r)n$. This process is called "restriction of scalars". In particular, it applies when f is the canonical morphism from R to R/I.

(c) Suppose $g : M \longrightarrow N$ is an R/I-module morphism. Then g is an abelian group morphism and $g\big((r+I)m\big) = (r+I)g(m)$. However, from the definitions in (a) and (b), we see that the last equation is the same as saying $g(rm) = rg(m)$. So g is an R/I-module morphism if and only if g is an R-module morphism.

3. Let F be a field and let $R = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \;\middle|\; a,b,c \in F \right\}$ be the ring of lower triangular 2×2 matrices over F. Put

$$A = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \;\middle|\; a,b \in F \right\},\quad B = \left\{ \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \;\middle|\; b \in F \right\} \text{ and } C = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \;\middle|\; c \in F \right\}.$$

   (a) Show that B, C and A/B are simple left R-modules.
   (b) Show that $B \cong C$ while B is not isomorphic to A/B.

Solution

(a) B is clearly closed under addition of matrices; also $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ cb_1 & 0 \end{bmatrix}$, so B is closed under the scalar multiplication. So B is a left ideal of R. We shall show that B is a minimal left ideal. Suppose $0 \neq K \leq B$ is an ideal of R. Then there exists $\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$ in K

with $b \neq 0$. Then $\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ xb^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \in K$ for all x. So $K = B$. So B is a simple left R-module.

Similarly, C is closed under addition and scalar multiplication and so is a left ideal of R. If $\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \in C$ with $c \neq 0$ then $\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & xc^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$; so C is a minimal left ideal of R; so C is a simple left R-module.

One easily sees that A is a left ideal of R; so A is a left R-module with B as a submodule. So A/B is the quotient module. Every element of A/B can be written as $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + B$. Assume $B < K \leq A$ for some ideal K of R. Then there is some $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ in K with $a \neq 0$. Then

$\begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} + B = \begin{bmatrix} a_1 a^{-1} & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + B \right) \in K/B$. So $A/B = K/B$. So A/B is simple.

(b) Define $f : B \longrightarrow C$ by $f \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$; this clearly preserves addition of matrices; also

$$f\left(\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b_1 & 0 \end{bmatrix} \right) = f\left(\begin{bmatrix} 0 & 0 \\ cb_1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & cb_1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b_1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} f\left(\begin{bmatrix} 0 & 0 \\ b_1 & 0 \end{bmatrix} \right).$$

So f is a left R-module morphism. It is clearly bijective. So $B \cong C$.

Assume $g : B \longrightarrow A/B$ is any left R-module morphism. Put $g \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + B$. Then

$$g \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix} = g\left(\begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix} g\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix} \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + B \right) = 0 + B.$$

So g is the zero morphism. So certainly there is no isomorphism $g : B \longrightarrow A/B$.

4. If e is an idempotent (that is, $e^2 = e$) element of the ring $\mathrm{End}_R(M)$ for some left R-module M then $M = \ker(e) \oplus \mathrm{im}(e)$.

Solution

Assume x is in $\ker(e) \cap \mathrm{im}(e)$. Then $e(x) = 0$ and $x = e(y)$ for some y in M. Then

$$x = e(y) = e^2(y) = e(e(y)) = e(x) = 0.$$

It follows that the sum $\ker(e) + \mathrm{im}(e)$ is direct. Moreover, every x in M can be written as $x = (x - e(x)) + e(x) \in \ker(e) + \mathrm{im}(e)$ since $e(x - e(x)) = e(x) - e(e(x)) = e(x) - e(x) = 0$ implies $x - e(x)$ is in $\ker(e)$ and certainly $e(x)$ is in $\mathrm{im}(e)$. So $M = \ker(e) \oplus \mathrm{im}(e)$.