# Analysis of an efficient construction of the reals

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July 9, 2002

The real numbers are traditionally constructed step by step from the natural, integer and rational numbers. The naturals are built from the five Peano postulates, the integers as equivalence classes of ordered pairs of naturals, the rational numbers as equivalence classes of ordered pairs of integers and the reals as equivalence classes of fundamental rational sequences.

This paper will present an efficient construction of the reals. We skip rational numbers and construct the reals directly from the set of integer numbers. Our investigation was motivated by Prof. Ross Street who explained this idea in a short paper [1] published in 1985.

## 1 Motivation

Notice that a real number $\alpha$ determines a function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = \text{i.p}(\alpha n)$, where "i.p" denotes "integer part".

Then $f(n)/n \to \alpha$ as $n \to \infty$ and $|f(m+n) - f(m) - f(n)| \leq 3$.

From that motivation, we attempt to construct the real number system directly from the set of integers by quasi-homomorphism functions in $\mathbb{Z}^{\mathbb{Z}}$.

First of all, some explanation about bounded and quasi-homomorphism function is needed.

Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ and the set $\mathbb{Z}^{\mathbb{Z}}$ of all functions from $\mathbb{Z}$ to $\mathbb{Z}$. This set is pointwise additive, i.e $(f+g)(x) = f(x) + g(x)$

[1] R. Street, *An efficient construction of the real numbers*, Gazette Australian Math Society. **12** (1985) 57-58

**Definition 1.** *A function $u : \mathbb{Z} \to \mathbb{Z}$ is bounded when its image is finite, i.e when exists $k \in \mathbb{N}$ such that $|u(x)| \leq k \ \forall x \in \mathbb{Z}$*

**Definition 2.** *Define $f \in \mathbb{Z}^{\mathbb{Z}}$ to be a quasi-homomorphism when $f(x+y) - f(x) - f(y)$ is bounded as a function of $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ i.e $|f(x+y) - f(x) - f(y)| \leq k$ for some $k \in \mathbb{N}$ and $\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}$*

**Theorem 1.** *The quasi-homomorphisms form a subgroup $qh(\mathbb{Z}, \mathbb{Z})$ of $\mathbb{Z}^{\mathbb{Z}}$*

*Proof.* Let $f, g$ be quasi-homomorphisms, then:

$$\begin{aligned} |f(x+y) - f(x) - f(y)| &\leq c_1 \\ |g(x+y) - g(x) - g(y)| &\leq c_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow |f(x+y) + g(x+y) - f(x) - g(x) - f(y) - g(y)| &\leq c_1 + c_2 \\ |(f+g)(x+y) - (f+g)(x) - (f+g)(y)| &\leq c_1 + c_2 \end{aligned}$$

Therefore $f + g$ is also a quasi-homomorphism.
Let $n \in \mathbb{Z}$. Then

$$\begin{aligned} n|f(x+y) - f(x) - f(y)| &\leq |n|c_1 \\ |nf(x+y) - nf(x) - nf(y)| &\leq |n|c_1 \end{aligned}$$

i.e $nf$ is also a quasi-homomorphism. $\square$

**Theorem 2.** *The bounded functions form a subgroup of $qh(\mathbb{Z}, \mathbb{Z})$*

*Proof.* Let $f, g$ be bounded functions, then

$$\begin{aligned} |f(x)| &\leq k_1 \quad \forall x \in \mathbb{Z} \\ |g(x)| &\leq k_2 \quad \forall x \in \mathbb{Z} \\ \Rightarrow |f(x) + g(x)| &\leq k_1 + k_2 \\ |(f+g)(x)| &\leq k_1 + k_2 \end{aligned}$$

Therefore $f + g$ is bounded.
Let $n \in \mathbb{Z}$, then:

$$\begin{aligned} n|f(x)| &\leq |n|k_1 \\ |nf(x)| &\leq |n|k_1 \end{aligned}$$

So $nf$ is bounded.

We should still show all bounded functions are quasi-homomorphisms.
Let $f$ be a bounded function, then

$$\begin{aligned} |f(x)| \leq k_1,\ |f(y)| &\leq k_1,\ |f(x+y)| \leq k_1 \\ \Rightarrow |f(x+y) - f(x) - f(y)| &\leq |f(x+y)| + |f(x)| + f(y)| \\ &\leq 3k_1 \end{aligned}$$

$\square$

**Definition 3 (Equivalence).** *Two functions $f, g \in \mathbb{Z}^{\mathbb{Z}}$ can be thought of as equivalent, denoted by $f \sim g$, when $\exists r \in \mathbb{N}$ such that $|f(m) - g(m)| \leq r\ \forall m$*

*Note* 1. It is easy to see that all bounded functions are equivalent.

*Proof.* Let $f, g$ be bounded functions, then $|f(m)| \leq k_1$, and $|g(m)| \leq k_2$

$$\begin{aligned} \Rightarrow |f(m) - g(m)| &\leq |f(m)| + |g(m)| \\ &\leq k_1 + k_2 \end{aligned}$$

$\square$

**Definition 4.** *Let us define an abelian group $\mathcal{R}$ by*

$$\mathcal{R} = qh(\mathbb{Z}, \mathbb{Z})/\sim$$

*and define $[f]$ to represent the equivalence class of all quasi-homomorphisms $\sim\ f$*

*Note* 2. $\mathcal{R}$ is an abelian group since it is the quotient of an abelian group by a subgroup.

In order to conclude that $\mathcal{R}$ is the field of real numbers, we have to examine whether it satisfises the Field Axioms, Order Axioms and Completeness properties.

## 2 Satisfaction of the Field Axioms

Since $\mathcal{R}$ is an abelian group under addition, it satisfies five axioms of addition. What we need to point out is that $[f] + [g] = [f + g]$, the definition of additive identity and $-[f] = [-f]$

**Definition 5 (Additive identity).** *Define $0 : \mathbb{Z} \to \mathbb{Z}$ given by $0(x) = 0$ for all $x \in \mathbb{Z}$. It is easy to see that $0(x)$ is a quasi-homomorphism, hence, $[0] \in \mathcal{R}$ and it is called an additive identity in $\mathcal{R}$.*

*Note* 3. $[0]$ represents all bounded functions since $|g(x) - 0(x)| \leq k \Leftrightarrow |g(x)| \leq k$. Therefore, $g \sim 0$, i.e $[g] = [0]$

**Definition 6 (Additive inverse).** *For $f \in \mathcal{R}$, define $-f$ such that $-f(x) = -(f(x))$. $[-f]$ is the called the additive inverse.*

Before proceeding to multiplicative structure, we define $[f][g]$ to be equal to $[g_o f]$ (where $g_o f$ is the composite of $f$ and $g$). To examine the satisfaction of the definition, we let $f' \in [f], g' \in [g]$ then prove $g'_o f' \sim g_o f$.

*Proof.* We have $|(g'_o f')(x) - (g_o f)(x)| = |g'(f'(x)) - g(f(x))|$.
Since, $f' \sim f$, $|f'(x) - f(x)| \leq k$.
Hence we write $f'(x) = f(x) + h(x), \quad |h(x)| \leq k$.

$$\begin{aligned} |g'(f'(x)) - g(f(x))| &= |g'(f'(x)) - g(f'(x)) + g(f'(x)) - g(f(x))| \\ &\leq |g'(f'(x)) - g(f'(x))| + |g(f'(x)) - g(f(x))| \\ &\leq k_2 + |g(f(x) + h(x)) - g(f(x))| \\ &\leq k_2 + |g(h(x))| + |g(f(x) + h(x)) - g(f(x)) - g(h(x))| \\ &\leq k_3 \end{aligned}$$

Note: since $|h(x)| \leq k$, we have $g(h(x))$ bounded.
Hence, $g'_o f' \sim g_o f$, i.e $[g'_o f'] = [g_o f]$ □

<**M1**>**(Multiplicative closure)** For every $[f], [g] \in \mathcal{R}$, we have $[f][g] \in \mathcal{R}$.

> *Proof.* $[f][g] = [g_o f] =$ all functions $\sim g(f(x))$. Since $f(x) \in \mathbb{Z}$, $g(f(x)) \in qh(\mathbb{Z}, \mathbb{Z})$
> Therefore, $[g_o f] \in \mathcal{R}$ □

<**M2**>**(Multiplicative associativity)** For every $[f], [g], [h] \in \mathcal{R}$, we have $[f]([g][h]) = ([f][g])[h]$.

> *Proof.* We need to show $[f_o(g_o h)] = [(f_o g)_o h]$
>
> $$\begin{aligned} |(f_o(g_o h)(x) - ((f_o g)_o h)(x)| &= |f(g(h(x))) - ((f_o g)(h(x)))| \\ &= |f(g(h(x))) - f(g(h(x)))| \\ &= 0 < k \end{aligned}$$
>
> Therefore $f_o(g_o h) \sim (f_o g)_o h$, i.e $[f_o(g_o h)] = [(f_o g)_o h]$ □

<**M3**>**(Multiplicative commutativity)** For every $[f], [g] \in \mathcal{R}$, we have $[f][g] = [g][f]$.

*Proof.* Let $[f] \in qh(\mathbb{Z}, \mathbb{Z})$, then $\forall m, n \in \mathbb{Z}, |f(m+n)-f(m)-f(m)| \leq k$
If $m \geq 0$, we have $(m-1)$ equations as follows:

$$\begin{aligned}
|f(n+n) - f(n) - f(n)| &\leq k \\
|f(2n+n) - f(2n) - f(n)| &\leq k \\
&\vdots \\
|f((m-1)n+n) - f((m-1)n) - f(n)| &\leq k
\end{aligned}$$

Adding these gives $|f(mn) - mf(n)| \leq (m-1)k$
If $m < 0$, we have $|m| + 1$ equations:

$$\begin{aligned}
|f(0+n) - f(0) - f(n)| &\leq k \\
|f(-n+n) - f(-n) - f(n)| &\leq k \\
|f(-2n+n) - f(-2n) - f(n)| &\leq k \\
&\vdots \\
|f(mn+n) - f(mn) - f(n)| &\leq k
\end{aligned}$$

Adding these gives

$$\begin{aligned}
|-f(mn) - |m|f(n)| &\leq (|m|+1)k \\
\Rightarrow |f(mn) + |m|f(n)| &\leq (|m|+1)k \\
\Rightarrow |f(mn) - mf(n)| &\leq (|m|+1)k
\end{aligned}$$

So $\forall m, |f(mn) - mf(n)| \leq (|m|+1)k$.
By symmetry, $|f(mn) - nf(m)| \leq (|n|+1)k$.
Therefore, $|nf(m) - mf(n)| \leq (|m|+|n|+2)k$.

Consider the inequality $|g(mn) - mg(n)| \leq (|m|+1)k$
Let $n = 1$:

$$\begin{aligned}
|g(m) - mg(1)| &\leq (|m|+1)k \\
\Rightarrow g(n) &\leq |ng(1)| + (|n|+1)k \\
g(n) &\leq (|n|+1)k_1 + (|n|+1)k \\
g(n) &\leq (|n|+1)k_2
\end{aligned}$$

Consider $|nf(m) - mf(n)| \leq (|m|+|n|+2)k$.

Let $m = g(n)$, then

$$\begin{aligned}
|nf(g(n)) - g(n)f(n)| &\leq ((|n|+1)k_2 + |n| + 2)k \\
&= (|n|+1)k_3 \\
|n(f_o g)(n) - g(n)f(n)| &\leq (|n|+1)k_3 \\
\Rightarrow |n(g_o f)(n) - f(n)g(n) &\leq (|n|+1)k_4 \\
\Rightarrow |n(g_o f)(n) - n(f_o g)(n)| &\leq (|n|+1)k_5 \\
|(g_o f)(n) - (f_o g)(n)| &\leq \left(\frac{|n|}{|n|} + \frac{1}{|n|}\right) k_5 \\
&= \left(1 + \frac{1}{|n|}\right) k_5 \\
&\leq 2k_5
\end{aligned}$$

Therefore $f_o g \sim g_o f$, so $[f_o g] = [g_o f]$, i.e $[g][f] = [f][g]$ □

<**M4**>**(Multiplicative identity)** For every $[f] \in \mathcal{R}$, there exists a unique element $[1] \in \mathcal{R}$, such that $[f][1] = [f]$.

**Definition 7 (Multiplicative identity).** *Define* $1 : \mathbb{Z} \to \mathbb{Z}$, *given by* $1(x) = x$. *It is easy to see that* $1 \in qh(\mathbb{Z}, \mathbb{Z})$, *i.e* $[1] \in \mathcal{R}$ *and is the set of all functions* $\sim 1$ . $[1]$ *is called the multiplicative identity.*

*Proof.* We have$[f][1] = [1_o f]$, so

$$\begin{aligned}
|(1_o f)(x) - f(x)| &= |1(f(x)) - f(x)| \\
&= |f(x) - f(x)| \\
&= 0 \leq k
\end{aligned}$$

Therefore, $1_o f \sim f$, so $[1_o f] = [f]$, i.e $[f][1] = [f]$.

□

Before proceeding to multiplication inverse, notice that $[0]$ cannot have an inverse. Let $[f] \neq [0]$, then $f$ is not bounded above (or below). Indeed, if $f$ is not bounded above then it is not bounded below.

*Proof.* Suppose that $f(x)$ is not bounded above as $x \to +\infty$. We will show that $f(x)$ is not bounded below as $x \to -\infty$. Since $f$ is quasi-homomorphism, we obtain:

$$\begin{aligned}
|f(x + (-x)) - f(x) - f(-x)| &\leq k \\
\Rightarrow |f(x) + f(-x)| &\leq k + |f(0)|
\end{aligned}$$

So $f(x) + f(-x)$ is bounded. However, $f(x)$ is not bounded above, hence, $f(-x)$ must not be bounded below.
$f(x)$ not bounded above as $x \to -\infty$ case follows similarly.
Therefore, if $f$ is not bounded above, then it is not bounded below. □

**Definition 8 (Left-bounded).** *$f$ is said to be left-bounded iff there exists $n \in \mathbb{Z}$ such that for all negative integer $m$, we have $|f(m)| < |n|$*

i.e, $f$ is not left-bounded iff for all n $\in \mathbb{Z}$, there exists a negative integer $m$ such that $|f(m)| \geq |n|$

**Definition 9 (Right-bounded).** *$f$ is said to be right-bounded iff $\exists n \in \mathbb{Z}$ such that for all $m \geq 0$ in $\mathbb{Z}$, we have $|f(m)| < |n|$*

i.e, $f$ is not right-bounded iff $\forall n \in \mathbb{Z}, \exists m \geq 0$ in $\mathbb{Z} : |f(m)| \geq |n|$

**<M5>(Multiplicative inverse)** For every $[f] \in \mathcal{R}$ such that $[f] \neq [0]$, there exists $[\bar{f}] \in \mathcal{R}$ such that $[f][\bar{f}] = [1]$

*Proof.* We are going to show that there exist such $[\bar{f}]$ and it is unique. Let $[f]$ be a non-zero element of $\mathcal{R}$. Then $f$ is not bounded.

i. If $f$ is not left-bounded

** For $n < 0$, let $\bar{f}(n) = m$ be the first negative integer such that $f(m) \leq n \Rightarrow f(m+1) > n$
Therefore, $f(\bar{f}(n)+1) > n \geq f(\bar{f}(n))$.
Since $f \in qh(\mathbb{Z}, \mathbb{Z})$, we have

$$|f(\bar{f}(n)+1) - f(\bar{f}(n)) - f(1)| \leq k$$

$$\Rightarrow f(\bar{f}(n)+1) \leq k + f(\bar{f}(n)) + f(1)$$

Hence,

$$\begin{aligned} n &< k + f(\bar{f}(n)) + f(1) \\ \Rightarrow n - f(\bar{f}(n)) &< k + f(1) \end{aligned}$$

Since $n - f(\bar{f}(n)) \geq 0, |n - f(\bar{f}(n))| < |k + f(1)| = k_1$
$\Rightarrow |1(n) - f(\bar{f}(n))| \leq k_2$, i.e $[\bar{f}][f] = [1]$

** For $n \geq 0$, let $\bar{f}(n) = m$ is the first negative integer such that $f(m) \geq n \Rightarrow f(m+1) < n$.
Therefore, $f(\bar{f}(n)+1) < n \leq f(\bar{f}(n))$. We have

$$|f(\bar{f}(n)+1) - f(\bar{f}(n)) - f(1)| \leq k$$

$$\Rightarrow f(\bar{f}(n)+1) \geq -k + f(\bar{f}(n)) + f(1)$$

Hence,

$$\begin{aligned} n &> -k + f(\bar{f}(n)) + f(1) \\ \Rightarrow f(\bar{f}(n)) - n &< k - f(1) \end{aligned}$$

Since $f(\bar{f}(n)) - n \geq 0, |f(\bar{f}(n)) - n| < |k - f(1)| = k_1$
$\Rightarrow |f(\bar{f}(n)) - 1(n)| \leq k_2$, i.e $[\bar{f}][f] = [1]$

ii. If $f$ is not right-bounded

** For $n < 0$, let $\bar{f}(n) = m$ is the first positive integer such that $f(m) \leq n \Rightarrow f(m-1) > n$.
So $f(\bar{f}(n) - 1) > n \geq f(\bar{f}(n))$.
Since $f \in qh(\mathbb{Z}, \mathbb{Z})$, we have

$$|f(\bar{f}(n) - 1) - f(\bar{f}(n)) - f(-1)| \leq k$$

$$\Rightarrow f(\bar{f}(n) - 1) \leq k + f(\bar{f}(n)) + f(-1)$$

Hence,

$$\begin{aligned} n &< k + f(\bar{f}(n)) + f(-1) \\ \Rightarrow n - f(\bar{f}(n)) &< k + f(-1) \end{aligned}$$

Since $n - f(\bar{f}(n)) \geq 0, |n - f(\bar{f}(n))| < |k + f(-1)| = k_1$
$\Rightarrow |1(n) - f(\bar{f}(n))| \leq k_2$, i.e $[\bar{f}][f] = [1]$

** For $n \geq 0$, let $\bar{f}(n) = m$ is the first positive integer such that $f(m) \geq n \Rightarrow f(m-1) < n$.
Therefore $f(\bar{f}(n) - 1) < n \leq f(\bar{f}(n))$. We have,

$$|f(\bar{f}(n) - 1) - f(\bar{f}(n)) - f(-1)| \leq k$$

$$\Rightarrow f(\bar{f}(n) - 1) \geq -k + f(\bar{f}(n)) + f(-1)$$

Hence,

$$\begin{aligned} n &> -k + f(\bar{f}(n)) + f(-1) \\ \Rightarrow f(\bar{f}(n)) - n &< k - f(-1) \end{aligned}$$

Since $f(\bar{f}(n)) - n \geq 0, |f(\bar{f}(n)) - n| < |k - f(-1)| = k_1$
$\Rightarrow |f(\bar{f}(n)) - 1(n)| \leq k_2$, i.e $[\bar{f}][f] = [1]$

Therefore, there exists an element $[\bar{f}]$ such that

$$[f][\bar{f}] = [1] = [\bar{f}][f].$$

Is $\bar{f}$ unique? Assume that $[\bar{f}']$ is also a multiplicative identity. Then

$$[f][\bar{f}'] = [1] = [f][\bar{f}].$$

Therefore, $|\bar{f}'(f(x)) - \bar{f}(f(x))| \leq k$
Let $y = f(x)$, we have $|\bar{f}'(y) - \bar{f}(y)| \leq k$. So $\bar{f}' \sim \bar{f}$.
Hence, $[\bar{f}'] = [\bar{f}]$ $\square$

<**D**>**(Distributive law)** For every $[f], [g], [h] \in \mathcal{R}$, we have

$$([g] + [h])[f] = [g][f] + [h][f]$$

*Proof.* We need to show $[f_o(g + h)] = [f_og] + [f_oh]$.
Since $f$ is quasi-homomorphism, we have

$$\begin{aligned}
|(f_o(g+h))(x) - ((f_og)(x) &+ (f_oh)(x))| \\
&= |f((g+h)(x)) - (f(g(x)) + f(h(x)))| \\
&= |f(g(x) + h(x)) - f(g(x)) - f(h(x))| \\
&\leq k
\end{aligned}$$

Therefore $f_o(g + h) \sim (f_og) + (f_oh)$
i.e $[f_o(g + h)] = [f_og] + [f_oh]$ □

# 3 Satisfaction of the Order Axioms

**Definition 10 (Positive).** *Call $[\alpha] \in \mathcal{R}$ positive when it can be represented by some quasi-homomorphism $f$ such that $f(n) \geq 0 \ \forall n \geq 0$*

Hence, we can alternatively define $[f] \in \mathcal{R}$ as positive iff

$$\exists k, f(n) \geq k \ \forall n \geq 0$$

**Definition 11.** *($[\alpha] \leq [\beta]$)*
*Say $[\alpha] \leq [\beta]$ when $[\beta - \alpha]$ is positive.*
$\Leftrightarrow \exists k, \beta(x) - \alpha(x) \geq k \ \forall x \geq 0$

*Note* 4. Say $|\beta(x) - \alpha(x)| \leq k_1 \ \forall x \geq 0$
i.e $|(\beta - \alpha)(x)| \leq k_1$ or $|f(x)| \leq k_1 \ \forall x \geq 0$
What can we say about $x < 0$?. Infact,

$$\begin{aligned}
|f(x + (-x)) - f(x) - f(-x)| &\leq k_2 \\
\Rightarrow |f(0) - f(x) - f(-x)| &\leq k_2 \\
\Rightarrow |f(-x)| &\leq k_2 + |-f(0)| + |f(x)|
\end{aligned}$$

Letting $x \geq 0$ then $|f(x)| \leq k_1$, therefore $|f(-x)| \leq k_3$.
Hence, if $f$ is bounded for $x \geq 0$, also bounded for $x < 0$.

<**O1**> For every $[a], [b] \in \mathcal{R}$, either $[a] \leq [b]$ or $[b] \leq [a]$.

*Proof.* Assume neither of those is true, then

$$\forall k_1, \exists x \geq 0 : b(x) - a(x) < k_1$$

$$\Rightarrow \forall k_3, \exists x \geq 0 : a(x) - b(x) > k_3$$

and

$$\forall k_2, \exists x \geq 0 : a(x) - b(x) < k_2$$

Let $f(x) = a(x) - b(x)$, since $f$ is quasi-homomorphism, we have

$$|f(x+y) - f(x) - f(y)| \leq K \ \forall x, y \in \mathbb{Z}$$

i.e

$$-K \leq f(x+y) - f(x) - f(y) \leq K$$

Let $x = X$ be the smallest value of $x \geq 0$ such that

$$a(x) - b(x) = f(x) > 0$$

Let $x = Y$ be the smallest value of $x \geq 0$ such that

$$a(x) - b(x) = f(x) < -K$$

Therefore, $f(X) > 0, f(Y) < -K$, note that $X \neq Y$

i. Suppose that $X > Y$, then we can write $X = Y + z, 0 < z \leq X$.
** For $Y \neq 0$, we have $0 < z < X$.
So $f(Y+z) = f(X) > 0$ and $f(z) < 0$.
Therefore, $f(Y+z) - f(Y) - f(z) > 0 - (-K) - 0 = K$, (C!).

**For $Y = 0$, we have $f(Y+z) - f(Y) - f(z) = -f(Y) > K$, (C!).

ii. Similarly for $Y > X$ case, we can show that

$$f(X+z) - f(X) - f(z) < -K$$

(C!).

Therefore, either $[a] \leq [b]$ or $[b] \leq [a]$ $\forall [a], [b] \in \mathcal{R}$
Note that it is also easy to see $[a] \leq [a]$ (reflexive) $\square$

<**O2**> For all $[a], [b] \in \mathcal{R}$, if $[a] \leq [b]$ and $[b] \leq [a]$ then $[a] = [b]$

*Proof.* We have, $[a] \leq [b]$ then

$$\forall x \geq 0, \exists k_1, b[x] - a[x] \geq k_1.$$

$[b] \leq [a]$ then

$$\forall x \geq 0, \exists k_2, a[x] - b[x] \geq k_2.$$

Therefore,

$$\forall x \geq 0, |b(x) - a(x)| \leq \min(|k_1|, |k_2|)$$

So $f(x) = b(x) - a(x)$ is bounded for all $x \geq 0$.
From note above, we have $f(x)$ is also bounded for all $x < 0$. Hence, $f(x)$ is bounded for all $x$, i.e $[f] = [0]$.
Therefore, $[a] = [b]$. $\square$

Hence, we have definition for strict inequality.

**Definition 12.** *For every* $[a], [b] \in \mathcal{R}$*, say* $[a] < [b]$ *iff* $[a] \leq [b]$ *and* $[a] \neq [b]$

<**O3**>**(Transitive)** For all $[a], [b], [c] \in \mathcal{R}$, if $[a] \leq [b]$ and $[b] \leq [c]$ then $[a] \leq [c]$

*Proof.* $\forall x \geq 0$, we have:

$$\exists k_1, b(x) - a(x) \geq k_1$$

$$\exists k_2, c(x) - b(x) \geq k_2$$

Adding those, we have $c(x) - a(x) \geq k_1 + k_2 = k_3,\ \forall x \geq 0$.
Therefore $[a] \leq [c]$ □

<**O4**> For every $[a], [b], [c] \in \mathcal{R}$ satisfying $[a] \leq [b]$, we have $[a]+[c] \leq [b]+[c]$

*Proof.* We have $[a] \leq [b]$ so

$$\begin{aligned} \forall x \geq 0, \exists k : b(x) - a(x) &\geq k \\ \forall x \geq 0, \exists k : b(x) + c(x) - c(x) - a(x) &\geq k \\ \Rightarrow \forall x \geq 0, \exists k : (b+c)(x) - (a+c)(x) &\geq k \end{aligned}$$

Therefore, $\Rightarrow [a] + [c] \leq [b] + [c]$ □

<**O5**> For every $[a], [b], [c] \in \mathcal{R}$ satisfying $[a] \leq [b]$, and $[c]$ is positive, we have $[c][a] \leq [c][b]$

*Proof.* We need to show $\forall x \geq 0, \exists K : b(c(x)) - a(c(x)) \geq K$
We have $[c]$ is positive so $\forall x \geq 0, \exists k_c : c(x) - k_c \geq 0$
Furthermore, $\forall x \geq 0\ \exists k : b(x) - a(x) \geq k$. Since $c(x) - k_c \geq 0$, we have

$$b(c(x) - k_c) - a(c(x) - k_c) \geq k$$

Since $b, a \in qh(\mathbb{Z}, \mathbb{Z})$, we obtain

$$|b(c(x) - k_c) - b(c(x)) - b(-k_c)| \leq k_b$$

$$\Rightarrow b(c(x) - k_c) \leq k_b + b(c(x)) + b(-k_c)$$

and

$$|a(c(x) - k_c) - a(c(x)) - a(-k_c)| \leq k_a$$

$$\Rightarrow a(c(x) - k_c) \geq -k_a + a(c(x)) + a(-k_c)$$

Therefore,

$$k_b + b(c(x)) + b(-k_c) - (-k_a + a(c(x)) + a(k_c)) \geq k,\ \forall x \geq 0$$

$$\Rightarrow b(c(x)) - a(c(x)) \geq k - k_b - b(-k_c) - k_a + a(-k_c) = K\ \forall x \geq 0$$

Hence, $[c][a] \leq [c][b]$ □

## 4 Completeness of $\mathcal{R}$

In order to show $\mathcal{R}$ is a complete ordered field, we need to show it has Archimedean property and every subset of $\mathcal{R}$ which has an upper bound has the least upper bound.

**Definition 13 (Natural number in $\mathcal{R}$).** *A natural number $[n]$(or $n$ for brief) is define to be a class of all function equivalent to an $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = nx$*

Does this definition satisfy all property of natural number? Indeed, it is easy to see that $n(x)+1(x) = nx+x = (n+1)x = (n+1)(x)$, so $[n+1]$ is the successor of $[n]$. It is also clear that $[m]+[n] = [m+n]$ is also a natural number. Hence, $\mathcal{R}$ contains all natural numbers (and integers, similarly) and we can use natural numbers in addition and multiplication as other $\mathcal{R}$ numbers.

Let $n$ be a natural number. Since $[n] \in \mathcal{R}$, there exists $[\bar{n}]$ such that $[n][\bar{n}] = [1]$. We denote $[\bar{n}]$ by $\dfrac{1}{n}$ (which is a rational number if we are allowed to know rational numbers).

**Theorem 3 (Archimedean property).** *For every $[a] \in \mathcal{R}$, there exist a natural number $n$ such that $[a] < n$*

*Proof.* As mentioned elsewhere, with $m, n \in \mathbb{N}$, we have

$$f(mn) - mf(n) \leq (m-1)k$$

Let $n = 1$, then

$$f(m) - mf(1) \leq (m-1)k < mk$$

Therefore, $f(m) < (f(1)+k)m = p \in \mathbb{N}\ \forall m \in \mathbb{N}$. □

**Theorem 4.** *Every subset of $\mathcal{R}$ which has an upper bound has a least upper bound.*

Before proving the theorem above, we need to show that a Cauchy sequence, which satisfies $|a_m - a_n| < \dfrac{1}{m} + \dfrac{1}{n}$ when $n$ and $m$ are large enough, converges.

*Proof.* Let $l(r) = f_r(r)$ with $[f_r]$ is Cauchy sequence satisfying the property above. If we can show that $l$ is quasi-homomorphism then $f_r$ does converge. Since $f$ is quasi-homomorphism, we have

$$\begin{aligned}
|l(m+n) - l(m) - l(n)| &= |f_{m+n}(m+n) - f_m(m) - f_n(n)| \\
&\leq |f_{m+n}(m) + f_{m+n}(n) - f_m(m) - f_n(n)| + k_{m+n} \\
&\leq |f_{m+n}(m) - f_m(m)| + |f_{m+n}(n) - f_n(n)| + k_{m+n} \\
&\leq \frac{1}{m+n} + \frac{1}{m} + \frac{1}{m+n} + \frac{1}{n} + k_{m+n} \leq k_{m+n} + 4
\end{aligned}$$

Since $k_{m+n}$ depends on $m$ and $n$, we need to show that $k_{m+n}$ can be replaced by a constant.
Suppose $f : \mathbb{N} \to \mathbb{N}$ is a quasi-homomorphism with

$$|f(m+n) - f(m) - f(n)| \leq k \text{ for all } m, n \in \mathbb{N}. \tag{1}$$

As mentioned eslewhere, for all $m \in \mathbb{N}$ , we have

$$|f(mn) - mf(n)| \leq (m-1)k \text{ for all } n \in \mathbb{N}. \tag{2}$$

and for all $m, n \in \mathbb{N}$

$$|nf(m) - mf(n)| \leq (m+n-2)k. \tag{3}$$

From (2), let $n = 1$, we have $f(m) - mf(1) \leq (m-1)k \leq mk$ so that

$$f(m) \leq (f(1) + k)m \ \forall m \in \mathbb{N}. \tag{4}$$

So $f(1) + k$ is an element of the set

$$S_f = \{u \in \mathbb{N} |\ f(m) \leq um \text{ for almost all } m \in \mathbb{N}\}$$

[Here "almost" means "all but a finite number".]

Since $\mathbb{N}$ is well-ordered, $S_f$ has a first element which we denote by $\lceil f \rceil \in \mathbb{N}$.
For $r \in \mathbb{N}$, we define the quasi-homomorphism $rf : \mathbb{N} \to \mathbb{N}$ by $(rf)(m) = rf(m)$ as usual. Define

$$\bar{f} : \mathbb{N} \to \mathbb{N} \text{ by } \bar{f}(r) = \lceil rf \rceil. \tag{5}$$

Then

$$rf(m) \leq \bar{f}(r)m \text{ for almost all } m \in \mathbb{N} \tag{6}$$

and

$$rf(m) \leq um \text{ for almost all } m \in \mathbb{N} \Rightarrow \bar{f}(r) \leq u. \tag{7}$$

By (6) and (3), we have

$$\bar{f}(r)m \geq rf(m) \geq mf(r) - (r+m-2)k.$$

So $\bar{f}(r) - f(r) \geq -\left(\dfrac{r+m-2}{m}k\right)$ for almost all $m \in \mathbb{N}, m > 0$.
Letting $m \to \infty$, we see that

$$\bar{f}(r) - f(r) \geq -k. \tag{8}$$

From (3) we have

$$\begin{aligned} rf(m) &\leq mf(r) + (m+r-2)k \\ &\leq (f(r)+k)m + (r-2)k \\ &\leq (f(r)+k+1)m \quad \text{for } m \geq k(r-2). \end{aligned}$$

By (7),

$$\bar{f}(r) \leq f(r) + k + 1. \tag{9}$$

So $-k \leq \bar{f}(r) - f(r) \leq k+1$. Hence $f$ and $\bar{f}$ are equivalent. We also need to show $\bar{f}$ is a quasi-homomorphism. In fact, for almost all $m \in \mathbb{N}$, we have

$$(r+s)f(m) = rf(m) + sf(m) \overset{(6)}{\leq} \bar{f}(r)m + \bar{f}(s)m.$$

By (7), this implies

$$\bar{f}(r+s) \leq \bar{f}(r) + \bar{f}(s). \tag{10}$$

On the other hand,

$$\begin{aligned}
(\bar{f}(r+s) - n\bar{f}(s))m &\overset{(6)}{\geq} (r+s)f(m) - \bar{f}(s)m \\
&\overset{(9)}{\geq} (r+s)f(m) - (f(s) + k + 1)m \\
&= rf(m) - (k+1)m \text{ for almost all } m.
\end{aligned}$$

By (7), we get

$$\bar{f}(r) \leq \bar{f}(r+s) - \bar{f}(s) + k + 1. \tag{11}$$

Combining (10) and (11), we get

$$0 \leq \bar{f}(r) + \bar{f}(s) - \bar{f}(r+s) \leq k+1. \tag{12}$$

So $\bar{f}$ is a quasi-homomorphism.

Symmetrically, we can define a quasi-homomorphism $\underline{f}$ equivalent to $f$ satisfying

$$-(k+1) \leq \underline{f}(r) + \underline{f}(s) - \underline{f}(r+s) \leq 0. \tag{13}$$

Define $g : \mathbb{N} \to \mathbb{N}$ by

$$g(m) = \left[\frac{\underline{f}(m) + \bar{f}(m)}{2}\right].$$

Then $g \sim f$ and

$$-\frac{k+3}{2} = -\frac{k+1}{2} - 1 \leq g(r) + g(s) - g(r+s) \leq \frac{k+1}{2} + 1 = \frac{k+3}{2}.$$

If $k \leq \dfrac{k+3}{2}$ then $k \leq 3$.
If not, by repeating this argument we can always find a representative quasi-homomorphism with $f(0) = 0$ and $k \leq 3$.
Come back to limit of Cauchy sequence, we have

$$|l(m+n) - l(m) - l(n)| \leq k_{m+n} + 4 \leq 3 + 4 = 7.$$

Therefore, $l$ is quasi-homomorphism and hence, $[f_r]$ is convergent. $\square$

*Proof.* (Theorem 4, see also Exersice 8.1 [2])
Let $S$ be a non empty and bounded above subset of $\mathcal{R}$. We need to show $S$ has least upper bound.
Note that from Archimedian property, we can find $n \in \mathbb{Z}$ such that $2x < n$, i.e, $x < \frac{n}{2}$.
Let $[a_1]$ be the smallest integer upper bound of $S$.
Choose $[a_2]$ as the smallest half integer upper bound of $S$.
Thus, either $[a_1] = [a_2]$ or $[a_1] = \left[a_2 + \frac{1}{2}\right]$, so

$$[0] \leq [a_1 - a_2] \leq \left[\frac{1}{2}\right]$$

Chose $[a_3]$ is the smallest quarter integer upper bound of $S$.
So either $[a_3] = [a_2]$ or $[a_3] = \left[a_2 - \frac{1}{4}\right]$. Therefore,

$$[0] \leq [a_2 - a_3] \leq \left[\frac{1}{4}\right]$$

Similarly, we have

$$[0] \leq [a_3 - a_4] \leq \left[\frac{1}{2^3}\right]$$

$$\vdots$$

$$[0] \leq [a_n - a_{n+1}] \leq \left[\frac{1}{2^n}\right]$$

$$\vdots$$

$$[0] \leq [a_{m-1} - a_m] \leq \left[\frac{1}{2^{m-1}}\right]$$

$\Rightarrow [0] \leq [a_n - a_m] \leq \left[\frac{1}{2^n}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\right] = \left[\frac{1}{2^{n-1}}\right] \leq \left[\frac{1}{n}\right]$
$\Rightarrow |[a_n] - [a_m]| < \frac{1}{n} + \frac{1}{m}$. So $a_n$ is a monotone decreasing Cauchy sequence of $\mathcal{R}$ numbers satisfying $|a_n - a_m| < \frac{1}{n} + \frac{1}{m}$ as $n, m$ large enough. Hence, $a_n$ converge to a limit, called $[l]$. We will show $[l]$ is the least upper bound of $S$.
Since $[l]$ is the limit of a sequence of upper bounds of $S$, $[l]$ is also an upper bound of $S$. If $[l]$ is not the least upper bound then there exists a term $[a_z]$ of the sequence such that $[a_z] < [l]$. However, $a_n$ is a monotone sequence, thus $a_n \leq [l]\ \forall n$, (C!).
Therefore, $S$ has a least upper bound.
Since $\mathcal{R}$ is an ordered field, the least upper bound is unique. $\square$

[2] R. P. Boas, Jr, *A Primer of Real Function*, The Corus Mathematic Monographs, **13**, MAA, Second Ed, 1972

## 5 Conclusion

We have shown that $\mathcal{R}$ is a complete ordered field. Therefore, we can conclude that $\mathcal{R}$ is the field of real numbers. This construction can be thought of as more efficient because we by pass the developtment of the rationals, i.e do not have to build and prove a range of definitions, theorems and properties of rationals.