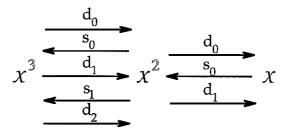
## **EVERY FACTORIZATION SYSTEM IS ORTHOGONAL**

**Notation** Each category X determines a simplicial category



where  $s_0: \mathcal{X} \longrightarrow \mathcal{X}^2$  takes  $X \in \mathcal{X}$  to  $1_X: X \longrightarrow X$ , where  $d_0, d_1: \mathcal{X}^2 \longrightarrow \mathcal{X}$  take  $f: A \longrightarrow B$  in  $\mathcal{X}^2$  to B, A, respectively, where  $s_0, s_1: \mathcal{X}^2 \longrightarrow \mathcal{X}^3$  take  $f: A \longrightarrow B$  to the composable pairs  $(1_A: A \longrightarrow A, f: A \longrightarrow B)$ ,  $(f: A \longrightarrow B, 1_B: B \longrightarrow B)$ , respectively, and, where  $d_0, d_1, d_2: \mathcal{X}^3 \longrightarrow \mathcal{X}^2$  take the composable pair  $(f: A \longrightarrow B, g: B \longrightarrow C)$  in  $\mathcal{X}^3$  to  $g: B \longrightarrow C$ ,  $g \circ f: A \longrightarrow B$ ,  $f: A \longrightarrow B$ , respectively. Recall that we have adjunctions  $d_0 \dashv s_0 \dashv d_1: \mathcal{X}^2 \longrightarrow \mathcal{X}$  involving a unit  $\eta: 1 \Rightarrow s_0 \circ d_0$  and a counit  $\epsilon: s_0 \circ d_1 \Rightarrow 1$ .

For functors  $P: \mathcal{A} \longrightarrow \mathcal{C}$ ,  $Q: \mathcal{B} \longrightarrow \mathcal{C}$ , we write  $\mathcal{C}(P,Q): \mathcal{A}^{op} \times \mathcal{B} \longrightarrow Set$  for the functor given by  $\mathcal{C}(P,Q)(A,B) = \mathcal{C}(PA,QB)$ ; so  $\mathcal{C}(1_{\mathcal{C}},1_{\mathcal{C}})$ , or merely  $\mathcal{C}(1,1)$ , denotes the hom functor of  $\mathcal{C}$ .

We shall make use of the natural transformation  $\theta: \mathcal{X}(d_0,\ d_1) \longrightarrow \mathcal{X}^2(1,1)$  whose component  $\theta_{f,g}\colon \mathcal{X}(B,C) \longrightarrow \mathcal{X}^2(f,g)$ , for objects  $f:A \longrightarrow B$ ,  $g:C \longrightarrow D$  of  $\mathcal{X}^2$ , is given by  $\theta_{f,g}(w)=(\ w\circ f,g\circ w\ )$ . More conceptually,  $\theta$  is the following composite.

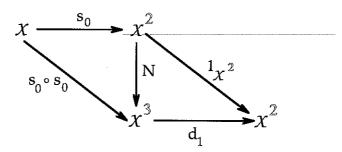
$$\mathcal{X}(d_0,d_1) \xrightarrow{s_0} \mathcal{X}^2(s_0 \circ d_0,s_0 \circ d_1) \xrightarrow{\mathcal{X}^2(\eta,\epsilon)} \mathcal{X}^2(1,1)$$

We say f is orthogonal to g when  $\theta_{f,g}$  is invertible.

**Proposition 1** For all objects  $(f: A \longrightarrow B, g: B \longrightarrow C)$ ,  $(f': A' \longrightarrow B', g': B' \longrightarrow C')$  of  $X^3$ , the following square is a pullback.

$$\begin{array}{c|c}
\mathcal{X}^{3}((f,g),(f',g')) & \xrightarrow{d_{1}} & \mathcal{X}^{2}(g \circ f, g' \circ f') \\
\downarrow d_{1} \circ d_{0} & & & \mathcal{X}^{2}((1,g),(f',1)) \\
\mathcal{X}(B,B') & \xrightarrow{\theta_{f,g'}} & \mathcal{X}^{2}(f,g')
\end{array}$$

**Definitions** A *decomposition* on a category X is a normalized splitting of  $d_1: X^3 \longrightarrow X^2$ ; that is, a functor  $N: X^2 \longrightarrow X^3$  such that the following diagram commutes.



Put  $\mathcal{E} = \{ f : A \longrightarrow B \mid d_0 \ N \ (f) \ invertible \}$ ,  $\mathcal{M} = \{ f : A \longrightarrow B \mid d_2 \ N \ (f) \ invertible \}$ . A factorization system is a decomposition with  $d_2 \ N \ (f) \in \mathcal{E}$  and  $d_0 \ N \ (f) \in \mathcal{M}$  for all arrows  $f : A \longrightarrow B$  in  $\mathcal{X}$ .

Suppose  $N: \mathcal{X}^2 \longrightarrow \mathcal{X}^3$  is a decomposition. We shall make use of the natural transformation  $\phi: \mathcal{X}(d_0,d_1) \longrightarrow \mathcal{X}^3(N,N)$  obtained by composing  $\theta$  with the effect of N on homs; explicitly, the component

$$\phi_{f\,,\,g}\colon\thinspace \mathcal{X}(B\,,C) {\:\longrightarrow\:} \mathcal{X}^{\mathbb{S}}\left(N(f)\,,N(g)\right)$$

is given by  $\phi_{f,g}(w) = (w \circ f, s \circ w \circ m, g \circ w)$  where we have put

$$N(f) = (e: A \longrightarrow I, m: I \longrightarrow B)$$
 and  $N(g) = (s: C \longrightarrow J, i: J \longrightarrow D)$ .

**Proposition 2** If  $f \in \mathcal{E}$  and  $g \in \mathcal{M}$  then  $\phi_{f,g}$  is invertible.

**Proof** The hypotheses mean m and s are both invertible. Any  $(u, x, v) : N(f) \longrightarrow N(g)$  has  $u = s^{-1} \circ x \circ e$ ,  $v = i \circ x \circ m^{-1}$ . The inverse of  $\phi_{f,g}$  takes (u, x, v) to  $s^{-1} \circ x \circ m^{-1}$ . Q.E.D.

**Proposition 3** Every factorization system  $N: X^2 \longrightarrow X^3$  is a fully faithful functor.

**Proof** Since  $N: \mathcal{X}^2(f,g) \longrightarrow \mathcal{X}^3(N(f),N(g))$  has  $d_1$  as a left inverse, it suffices to prove  $d_1$  is injective. The pullback of an injective function is injective so, by applying Proposition 1 to the objects N(f), N(g) of  $\mathcal{X}^3$ , we see that it suffices to prove  $\theta_{e,i}$  is injective. So it suffices to prove  $\phi_{e,i} = N \circ \theta_{e,i}$  is injective. But N is a factorization system, so  $e \in \mathcal{E}$  and  $i \in \mathcal{M}$ . The result now follows from Proposition 2. **Q.E.D.** 

**Corollary** For a factorization system, each  $f \in \mathcal{E}$  is orthogonal to each  $g \in \mathcal{M}$ .

**Proof** Propositions 2 and 3 and the equality  $\phi_{f,g} = N \circ \theta_{f,g}$  imply  $\theta_{f,g}$  is invertible. **Q.E.D.** 

Clearly both these classes include all identity arrows.

Ems compose 7 January 2000

(1) First recall the 2-cell aspect of the universal property of X2.

 $K \xrightarrow{u} X \xrightarrow{d_i} X$ 

If  $\rho$ ,  $\sigma$  are 2-cells such that  $d, u \stackrel{\rho}{\Longrightarrow} d, v$ 

Tul don It commutes

then there exists a unique 2-cell  $\omega$ ;  $u \Rightarrow v$  such that

 $d, \omega = e & do \omega = \pi$ .

Défine e: X2 by  $X^{2} = X^{2} = X^{2} = X^{2} = X^{2} = X^{2}$ Axion1 ne: je » j is invertible. dig sig ug

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ng then O is invertible. 3 Theorem Suppose (j, E, M) is a decomposition on X satisfying asions 1 & 2. For F: A -> X, fut  $\varphi_i = \left(A \xrightarrow{f} X \xrightarrow{g} X \xrightarrow{g} X\right)$ for i = 0,1,2. Then 90, 92 mvertible implies 9, mvertible.

Page S Proof By D, commutativity of  $d_1d_1 = d_1d_2$   $d_2 = d_1d_2$ implies ]! v;d, >do: X = X such that  $d_1 t = \lambda d_2 : d_i d_i \Longrightarrow d_i d_0$  $A d_0 c = 1 : d_0 d_1 = d_0 d_0$ . Thus we have commutativity in  $q = \epsilon d_i f$   $j d_i f$ [nd,f] III [ndof] dodf dotf = 1 dodof; the top square of this gives commitativity in  $d_i e d_i f = d_i d_i f = d_i d_i f$  $\lambda = d, f$   $\lambda = 1$   $\lambda = 1$   $\lambda = 1$  $d_{ed}f = \int d_{i}f \xrightarrow{j \in f} \int d_{o}f \xrightarrow{\varphi^{-1}} d_{o}d_{z}f$ .

tage4 So, by D, there exists a unique  $\sigma: ed, f \longrightarrow d_2f$ such that  $d_i \sigma = 1$  and  $d_o \sigma = q_o (j r f)$ . Thus we have commutativity in  $d_1ed_1f = \frac{d_1\sigma = 1}{}$ adif= Eedif  $\int \varepsilon d_2 f = \varphi_2$ jed,f > jdzf ned,f] µd2f doed,f doo dodzf jef jdof = and so  $\epsilon d, f = \frac{\beta \epsilon dif}{\beta dif} > jd, f$ ud, f = (udof)(jef)[(µed, f)-1

(Notice that we used less than Axiom 1: merely that ned, f is invertible.) By assom 2,  $\theta = \varphi_1 \cdot \varphi_2 \cdot j\sigma \cdot (\mu ed, f)$ is invertible. So jo is a split monic. From V and the invertibility of P2, we see that jo is split epic. So jo is invertible. So  $\varphi_1 = \Theta \cdot (\mu ed, f) \cdot (j\sigma) \cdot \varphi_2$ is invertible as required. QED