DISCRETE MATHEMATICS

W W L CHEN

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Chapter 3

THE NATURAL NUMBERS

3.1. Introduction

The set of natural numbers is usually given by

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

However, this definition does not bring out some of the main properties of the set $\mathbb{N}$ in a natural way. The following more complicated definition is therefore sometimes preferred.

DEFINITION. The set $\mathbb{N}$ of all natural numbers is defined by the following four conditions:
- (N1) $1 \in \mathbb{N}$.
- (N2) If $n \in \mathbb{N}$, then the number $n + 1$, called the successor of n, also belongs to $\mathbb{N}$.
- (N3) Every $n \in \mathbb{N}$ other than 1 is the successor of some number in $\mathbb{N}$.
- (WO) Every non-empty subset of $\mathbb{N}$ has a least element.

The condition (WO) is called the Well-ordering principle.

To explain the significance of each of these four requirements, note that the conditions (N1) and (N2) together imply that $\mathbb{N}$ contains $1, 2, 3, \ldots$. However, these two conditions alone are insufficient to exclude from $\mathbb{N}$ numbers such as 5.5. Now, if $\mathbb{N}$ contained 5.5, then by condition (N3), $\mathbb{N}$ must also contain $4.5, 3.5, 2.5, 1.5, 0.5, -0.5, -1.5, -2.5, \ldots$, and so would not have a least element. We therefore exclude this possibility by stipulating that $\mathbb{N}$ has a least element. This is achieved by the condition (WO).

3.2. Induction

It can be shown that the condition (WO) implies the Principle of induction. The following two forms of the Principle of induction are particularly useful.

PRINCIPLE OF INDUCTION (WEAK FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIW1) $p(1)$ is true; and
(PIW2) $p(n+1)$ is true whenever $p(n)$ is true.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

PROOF. Suppose that the conclusion does not hold. Then the subset

$$S = \{n \in \mathbb{N} : p(n) \text{ is false}\}$$

of $\mathbb{N}$ is non-empty. By (WO), S has a least element, n_0 say. If $n_0 = 1$, then clearly (PIW1) does not hold. If $n_0 > 1$, then $p(n_0 - 1)$ is true but $p(n_0)$ is false, contradicting (PIW2). ◯

PRINCIPLE OF INDUCTION (STRONG FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIS1) $p(1)$ is true; and
(PIS2) $p(n+1)$ is true whenever $p(m)$ is true for all $m \le n$.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

PROOF. Suppose that the conclusion does not hold. Then the subset

$$S = \{n \in \mathbb{N} : p(n) \text{ is false}\}$$

of $\mathbb{N}$ is non-empty. By (WO), S has a least element, n_0 say. If $n_0 = 1$, then clearly (PIS1) does not hold. If $n_0 > 1$, then $p(m)$ is true for all $m \le n_0 - 1$ but $p(n_0)$ is false, contradicting (PIS2). ◯

In the examples below, we shall illustrate some basic ideas involved in proof by induction.

EXAMPLE 3.2.1. We shall prove by induction that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \tag{1}$$

for every $n \in \mathbb{N}$. To do so, let $p(n)$ denote the statement (1). Then clearly $p(1)$ is true. Suppose now that $p(n)$ is true, so that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Then

$$1 + 2 + 3 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2},$$

so that $p(n+1)$ is true. It now follows from the Principle of induction (Weak form) that (1) holds for every $n \in \mathbb{N}$.

EXAMPLE 3.2.2. We shall prove by induction that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

for every $n \in \mathbb{N}$. To do so, let $p(n)$ denote the statement (2). Then clearly $p(1)$ is true. Suppose now that $p(n)$ is true, so that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Then

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}, \end{aligned}$$

so that $p(n+1)$ is true. It now follows from the Principle of induction (Weak form) that (2) holds for every $n \in \mathbb{N}$.

Example 3.2.3. We shall prove by induction that $3^n > n^3$ for every $n > 3$. To do so, let $p(n)$ denote the statement

$$(n \leq 3) \vee (3^n > n^3).$$

Then clearly $p(1), p(2), p(3), p(4)$ are all true. Suppose now that $n > 3$ and $p(n)$ is true. Then $3^n > n^3$. It follows that

$$\begin{aligned} 3^{n+1} &> 3n^3 = n^3 + 2n^3 > n^3 + 6n^2 = n^3 + 3n^2 + 3n^2 > n^3 + 3n^2 + 6n \\ &= n^3 + 3n^2 + 3n + 3n > n^3 + 3n^2 + 3n + 1 = (n+1)^3 \end{aligned}$$

(note that we are aiming for $(n+1)^3 = n^3 + 3n^2 + 3n + 1$ all the way), so that $p(n+1)$ is true. It now follows from the Principle of induction (Weak form) that $3^n > n^3$ holds for every $n > 3$.

Example 3.2.4. We shall prove by induction the famous De Moivre theorem that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta \tag{3}$$

for every $\theta \in \mathbb{R}$ and every $n \in \mathbb{N}$. To do so, let $\theta \in \mathbb{R}$ be fixed, and let $p(n)$ denote the statement (3). Then clearly $p(1)$ is true. Suppose now that $p(n)$ is true, so that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$

Then

$$\begin{aligned} (\cos\theta + \mathrm{i}\sin\theta)^{n+1} &= (\cos n\theta + \mathrm{i}\sin n\theta)(\cos\theta + \mathrm{i}\sin\theta) \\ &= (\cos n\theta\cos\theta - \sin n\theta\sin\theta) + \mathrm{i}(\sin n\theta\cos\theta + \cos n\theta\sin\theta) \\ &= \cos(n+1)\theta + \mathrm{i}\sin(n+1)\theta, \end{aligned}$$

so that $p(n+1)$ is true. It now follows from the Principle of induction (Weak form) that (3) holds for every $n \in \mathbb{N}$.

Example 3.2.5. Consider the sequence $x_1, x_2, x_3, \ldots$, given by $x_1 = 5$, $x_2 = 11$ and

$$x_{n+1} - 5x_n + 6x_{n-1} = 0 \qquad \text{whenever } n \geq 2. \tag{4}$$

We shall prove by induction that

$$x_n = 2^{n+1} + 3^{n-1} \tag{5}$$

for every $n \in \mathbb{N}$. To do so, let $p(n)$ denote the statement (5). Then clearly $p(1), p(2)$ are both true. Suppose now that $n \geq 2$ and $p(m)$ is true for every $m \leq n$, so that $x_m = 2^{m+1} + 3^{m-1}$ for every $m \leq n$. Then

$$\begin{aligned} x_{n+1} &= 5x_n - 6x_{n-1} = 5(2^{n+1} + 3^{n-1}) - 6(2^{n-1+1} + 3^{n-1-1}) \\ &= 2^n(10-6) + 3^{n-2}(15-6) = 2^{n+2} + 3^n, \end{aligned}$$

so that $p(n+1)$ is true. It now follows from the Principle of induction (Strong form) that (5) holds for every $n \in \mathbb{N}$.

Problems for Chapter 3

1. Prove by induction each of the following identities, where $n \in \mathbb{N}$:

   a) $\displaystyle\sum_{k=1}^{n}(2k-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

   b) $\displaystyle\sum_{k=1}^{n}k(k+2) = \frac{n(n+1)(2n+7)}{6}$

   c) $\displaystyle\sum_{k=1}^{n}k^3 = (1+2+3+\ldots+n)^2$

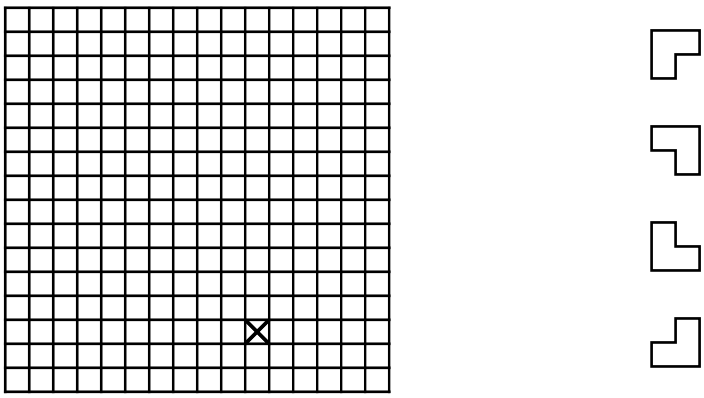
   d) $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)} = \frac{n}{n+1}$

2. Suppose that $x \neq 1$. Prove by induction that

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}.$$

3. Find the smallest $m \in \mathbb{N}$ such that $m! \geq 2^m$. Prove that $n! \geq 2^n$ for every $n \in \mathbb{N}$ satisfying $n \geq m$.

4. Consider a $2^n \times 2^n$ chessboard with one arbitrarily chosen square removed, as shown in the picture below (for $n = 4$):



   Prove by induction that any such chessboard can be tiled without gaps or overlaps by L-shapes consisting of 3 squares each as shown.

5. The sequence a_n is defined recursively for $n \in \mathbb{N}$ by $a_1 = 3$, $a_2 = 5$ and

$$a_n = 3a_{n-1} - 2a_{n-2} \qquad \text{whenever } n \geq 3.$$

   Prove that $a_n = 2^n + 1$ for every $n \in \mathbb{N}$.

6. For every $n \in \mathbb{N}$ and every $k = 0, 1, 2, \ldots, n$, the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

with the convention that $0! = 1$.

a) Show from the definition and without using induction that for every $k = 1, 2, \ldots, n$, we have

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

b) Use part (a) and the Principle of induction to prove the Binomial theorem, that for every $n \in \mathbb{N}$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

7. The "theorem" below is clearly absurd. Find the mistake in the "proof".
   - Theorem: Let $\ell_1, \ell_2, \ldots, \ell_n$ be $n \geq 2$ distinct lines on the plane, no two of which are parallel. Then all these lines have a point in common.
   - Proof: For $n = 2$, the statement is clearly true, since any 2 non-parallel lines on the plane intersect. Assume now that the statement holds for $n = k$, and let us now have $k + 1$ lines $\ell_1, \ell_2, \ldots, \ell_{k-1}, \ell_k, \ell_{k+1}$. By the inductive hypothesis, the k lines $\ell_1, \ell_2, \ldots, \ell_k$ (omitting the line ℓ_{k+1}) have a point in common; let us denote this point by x. Again by the inductive hypothesis, the k lines $\ell_1, \ell_2, \ldots, \ell_{k-1}, \ell_{k+1}$ (omitting the line ℓ_k) have a point in common; let us denote this point by y. The line ℓ_1 lies in both collections, so it contains both points x and y. The line ℓ_{k-1} also lies in both collections, so it also contains both points x and y. Now the lines ℓ_1 and ℓ_{k-1} intersect at one point only, so we must have $x = y$. Therefore the $k + 1$ lines $\ell_1, \ell_2, \ldots, \ell_{k-1}, \ell_k, \ell_{k+1}$ have a point in common, namely the point x. The result now follows from the Principle of induction.