

DISTRIBUTION OF PRIME NUMBERS

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Chapter 3

DIRICHLET SERIES

3.1. Convergence Properties

A Dirichlet series is a series of the type

$$(1) \quad F(s) = \sum_{n=1}^{\infty} f(n)n^{-s},$$

where $f : \mathbb{N} \rightarrow \mathbb{C}$ is an arithmetic function and $s \in \mathbb{C}$. We usually write $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$.

Our first task is to investigate the convergence properties of Dirichlet series.

THEOREM 3A. *Suppose that the series (1) converges for some $s \in \mathbb{C}$. Then there exist unique real numbers $\sigma_0, \sigma_1, \sigma_2$ satisfying $-\infty \leq \sigma_0 \leq \sigma_1 \leq \sigma_2 < \infty$ and such that the following statements hold:*

- (i) *The series (1) converges for every $s \in \mathbb{C}$ with $\sigma > \sigma_0$. Furthermore, for every $\epsilon > 0$, the series (1) diverges for some $s \in \mathbb{C}$ with $\sigma_0 - \epsilon < \sigma \leq \sigma_0$.*
- (ii) *For every $\eta > 0$, the series (1) converges uniformly on the set $\{s \in \mathbb{C} : \sigma > \sigma_1 + \eta\}$ and does not converge uniformly on the set $\{s \in \mathbb{C} : \sigma > \sigma_1 - \eta\}$.*
- (iii) *The series (1) converges absolutely for every $s \in \mathbb{C}$ with $\sigma > \sigma_2$. Furthermore, for every $\epsilon > 0$, the series (1) does not converge absolutely for some $s \in \mathbb{C}$ with $\sigma_2 - \epsilon < \sigma \leq \sigma_2$.*

EXAMPLE. The Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges absolutely for every $s \in \mathbb{C}$ with $\sigma > 1$ and diverges for every real $s < 1$. It follows that $\sigma_0 = \sigma_1 = \sigma_2 = 1$ in this case.

PROOF OF THEOREM 3A. Suppose that the series (1) converges for $s = s^* = \sigma^* + it^*$. Then $f(n)n^{-s^*} \rightarrow 0$ as $n \rightarrow \infty$, so that $|f(n)n^{-s^*}| = O(1)$, and so $|f(n)| = O(n^{\sigma^*})$. It follows that for every $s \in \mathbb{C}$ with $\sigma > \sigma^* + 1$, we have

$$|f(n)n^{-s}| = |f(n)n^{-\sigma}| = O(n^{\sigma^* - \sigma}),$$

so that the series (1) converges by the Comparison test. Now let

$$\sigma_0 = \inf\{u \in \mathbb{R} : \text{the series (1) converges for all } s \in \mathbb{C} \text{ with } \sigma > u\}$$

and

$$\sigma_2 = \inf\{u \in \mathbb{R} : \text{the series (1) converges absolutely for all } s \in \mathbb{C} \text{ with } \sigma > u\}.$$

Clearly (i) and (iii) follow, and $\sigma_0 \leq \sigma_2$. To prove (ii), let $\delta > 0$ and $\epsilon > 0$ be chosen. Then there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |f(n)|n^{-\sigma_2 - \delta} < \epsilon.$$

Hence

$$\sup \left\{ \left| \sum_{n=1}^N f(n)n^{-s} - \sum_{n=1}^{\infty} f(n)n^{-s} \right| : \sigma \geq \sigma_2 + \delta \right\} \leq \sum_{n=N+1}^{\infty} |f(n)|n^{-\sigma_2 - \delta} < \epsilon.$$

It follows that the series (1) converges uniformly on the set $\{s \in \mathbb{C} : \sigma \geq \sigma_2 + \delta\}$. Now let

$$\sigma_1 = \inf\{u \in \mathbb{R} : \text{the series (1) converges uniformly on } \{z \in \mathbb{C} : \sigma \geq u\}\}.$$

Clearly $\sigma_0 \leq \sigma_1 \leq \sigma_2 + \delta$. Since $\delta > 0$ is arbitrary, we must have $\sigma_0 \leq \sigma_1 \leq \sigma_2$. \circ

A simple consequence of uniform convergence is the following result concerning differentiation term by term.

THEOREM 3B. *For every $s \in \mathbb{C}$ with $\sigma > \sigma_1$, the series (1) may be differentiated term by term. In particular, $F'(s)$ exists and*

$$F'(s) = - \sum_{n=1}^{\infty} f(n)(\log n)n^{-s}.$$

3.2. Uniqueness Properties

Our next task is to prove the uniqueness theorem of Dirichlet series, a result of great importance in view of the applications we have in mind.

THEOREM 3C. *Suppose that*

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} g(n)n^{-s},$$

where $f : \mathbb{N} \rightarrow \mathbb{C}$ and $g : \mathbb{N} \rightarrow \mathbb{C}$ are arithmetic functions and $s \in \mathbb{C}$. Suppose further that there exists $\sigma_3 \in \mathbb{R}$ such that for every $s \in \mathbb{C}$ satisfying $\sigma \geq \sigma_3$, we have $F(s) = G(s)$. Then $f(n) = g(n)$ for every $n \in \mathbb{N}$.

It is clearly sufficient to prove the following special case.

THEOREM 3D. *Suppose that*

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s},$$

where $f : \mathbb{N} \rightarrow \mathbb{C}$ is an arithmetic function and $s \in \mathbb{C}$. Suppose further that there exists $\sigma_3 \in \mathbb{R}$ such that for every $s \in \mathbb{C}$ satisfying $\sigma \geq \sigma_3$, we have $F(s) = 0$. Then $f(n) = 0$ for every $n \in \mathbb{N}$.

PROOF. Since the series converges for $s = \sigma_3$, we must have $|f(n)| = O(n^{\sigma_3})$ for all $n \in \mathbb{N}$. Now let $\sigma \geq \sigma_3 + 2$. Then

$$(2) \quad \sum_{n=N}^{\infty} f(n)n^{-\sigma} = O\left(\sum_{n=N}^{\infty} n^{\sigma_3-\sigma}\right).$$

Note next that $y^{\sigma_3-\sigma}$ is a decreasing function of y , so that

$$(3) \quad \sum_{n=N}^{\infty} n^{\sigma_3-\sigma} = N^{\sigma_3-\sigma} + \sum_{n=N+1}^{\infty} n^{\sigma_3-\sigma} \leq N^{\sigma_3-\sigma} + \int_N^{\infty} y^{\sigma_3-\sigma} dy = O(N^{\sigma_3-\sigma+1}).$$

Combining (2) and (3), we see that for every $N \in \mathbb{N}$, we have

$$(4) \quad \sum_{n=N}^{\infty} f(n)n^{-\sigma} = O(N^{\sigma_3-\sigma+1}).$$

Using (4) with $N = 2$, we obtain, for $\sigma \geq \sigma_3 + 2$,

$$0 = F(\sigma) = f(1) + \sum_{n=2}^{\infty} f(n)n^{-\sigma} = f(1) + O(2^{\sigma_3-\sigma+1}) \rightarrow f(1) \quad \text{as } \sigma \rightarrow +\infty.$$

Hence $f(1) = 0$. Suppose now that $f(1) = f(2) = \dots = f(M-1) = 0$. Using (4) with $N = M+1$, we obtain, for $\sigma \geq \sigma_3 + 2$,

$$0 = F(\sigma) = f(M)M^{-\sigma} + \sum_{n=M+1}^{\infty} f(n)n^{-\sigma} = f(M)M^{-\sigma} + O((M+1)^{\sigma_3-\sigma+1}),$$

so that

$$0 = f(M) + O\left((M+1)^{\sigma_3+1} \left(\frac{M}{M+1}\right)^{\sigma}\right) \rightarrow f(M) \quad \text{as } \sigma \rightarrow +\infty.$$

Hence $f(M) = 0$. The result now follows from induction. \circ

3.3. Multiplicative Properties

Dirichlet series are extremely useful in tackling problems in number theory as well as in other branches of mathematics. The main properties that underpin most of these applications are the multiplicative aspects of these series.

THEOREM 3E. *Suppose that for every $j = 1, 2, 3$, we have*

$$F_j(s) = \sum_{n=1}^{\infty} f_j(n)n^{-s},$$

where $f_j : \mathbb{N} \rightarrow \mathbb{C}$ is an arithmetic function and $s \in \mathbb{C}$. Suppose further that for every $n \in \mathbb{N}$, we have

$$f_3(n) = \sum_{\substack{x,y \\ xy=n}} f_1(x)f_2(y) = \sum_{x|n} f_1(x)f_2\left(\frac{n}{x}\right) = \sum_{y|n} f_1\left(\frac{n}{y}\right)f_2(y).$$

Then

$$F_1(s)F_2(s) = F_3(s),$$

provided that $\sigma > \max\{\sigma_2^{(1)}, \sigma_2^{(2)}\}$, where, for every $j = 1, 2$, the series $F_j(s)$ converges absolutely for every $s \in \mathbb{C}$ with $\sigma > \sigma_2^{(j)}$.

PROOF. We have

$$\sum_{n=1}^N f_3(n)n^{-s} = \sum_{\substack{1 \leq x \leq N \\ 1 \leq y \leq N \\ xy \leq N}} f_1(x)x^{-s}f_2(y)y^{-s},$$

so that

$$\begin{aligned} \sum_{n=1}^N f_3(n)n^{-s} &- \sum_{x \leq \sqrt{N}} f_1(x)x^{-s} \sum_{y \leq \sqrt{N}} f_2(y)y^{-s} \\ &= \sum_{\sqrt{N} < x \leq N} f_1(x)x^{-s} \sum_{y \leq N/x} f_2(y)y^{-s} + \sum_{x \leq \sqrt{N}} f_1(x)x^{-s} \sum_{\sqrt{N} < y \leq N/x} f_2(y)y^{-s}. \end{aligned}$$

It follows that

$$\begin{aligned} (5) \quad &\left| \sum_{n=1}^N f_3(n)n^{-s} - \sum_{x \leq \sqrt{N}} f_1(x)x^{-s} \sum_{y \leq \sqrt{N}} f_2(y)y^{-s} \right| \\ &< \left(\sum_{x > \sqrt{N}} |f_1(x)|x^{-\sigma} \right) \left(\sum_{y=1}^{\infty} |f_2(y)|y^{-\sigma} \right) + \left(\sum_{x=1}^{\infty} |f_1(x)|x^{-\sigma} \right) \left(\sum_{y > \sqrt{N}} |f_2(y)|y^{-\sigma} \right). \end{aligned}$$

Suppose now that $\sigma > \max\{\sigma_2^{(1)}, \sigma_2^{(2)}\}$. Clearly

$$\sum_{x > \sqrt{N}} |f_1(x)|x^{-\sigma} \quad \text{and} \quad \sum_{y > \sqrt{N}} |f_2(y)|y^{-\sigma}$$

converge to 0 as $N \rightarrow \infty$. Furthermore, the series

$$\sum_{x=1}^{\infty} |f_1(x)|x^{-\sigma} \quad \text{and} \quad \sum_{y=1}^{\infty} |f_2(y)|y^{-\sigma}$$

are convergent. It follows that the right hand side of (5) converges to 0 as $N \rightarrow \infty$. On the other hand,

$$\sum_{x \leq \sqrt{N}} f_1(x)x^{-s} \quad \text{and} \quad \sum_{y \leq \sqrt{N}} f_2(y)y^{-s}$$

converge to $F_1(s)$ and $F_2(s)$ respectively as $N \rightarrow \infty$. The result follows. \circ

REMARK. Theorem 3E generalizes to a product of k Dirichlet series $F_1(s), \dots, F_k(s)$, where the general coefficient is

$$\sum_{\substack{x_1, \dots, x_k \\ x_1 \dots x_k = n}} f_1(x_1) \dots f_k(x_k).$$

In many applications, the coefficients $f(n)$ of the Dirichlet series will be given by various important arithmetic functions in number theory. We therefore study next some consequences when the function $f : \mathbb{N} \rightarrow \mathbb{C}$ is multiplicative.

THEOREM 3F. *Suppose that the function $f : \mathbb{N} \rightarrow \mathbb{C}$ is multiplicative. Then for every $s \in \mathbb{C}$ satisfying $\sigma > \sigma_2$, the series (1) satisfies*

$$F(s) = \prod_p \left(\sum_{h=0}^{\infty} f(p^h) p^{-hs} \right).$$

PROOF. By the Remark, if p_j is the j -th prime in increasing order, then

$$\prod_{j=1}^k \left(\sum_{h=0}^{\infty} f(p_j^h) p_j^{-hs} \right) = \sum_{n=1}^{\infty} \left(\sum_{\substack{h_1, \dots, h_k \\ p_1^{h_1} \dots p_k^{h_k} = n}} f(p_1^{h_1}) \dots f(p_k^{h_k}) \right) n^{-s}.$$

By the uniqueness of factorization, the inner sum on the right hand side contains at most one term. Hence

$$\prod_{j=1}^k \left(\sum_{h=0}^{\infty} f(p_j^h) p_j^{-hs} \right) = \sum_{n=1}^{\infty} \theta_k(n) f(n) n^{-s},$$

where

$$\theta_k(n) = \begin{cases} 1 & \text{if all the prime factors of } n \text{ are among } p_1, \dots, p_k, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} \prod_{j=1}^k \left(\sum_{h=0}^{\infty} f(p_j^h) p_j^{-hs} \right) - \sum_{n=1}^{\infty} f(n) n^{-s} &= \sum_{n=1}^{\infty} (\theta_k(n) - 1) f(n) n^{-s} \\ &= O \left(\sum_{n=k+1}^{\infty} |f(n)| n^{-\sigma} \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

The result follows. \circ

An arithmetic function $f : \mathbb{N} \rightarrow \mathbb{C}$ is said to be totally multiplicative or strongly multiplicative if $f(mn) = f(m)f(n)$ for every $m, n \in \mathbb{N}$.

THEOREM 3G. *Suppose that the function $f : \mathbb{N} \rightarrow \mathbb{C}$ is totally multiplicative. Then for every $s \in \mathbb{C}$ satisfying $\sigma > \sigma_2$, the series (1) satisfies*

$$F(s) = \prod_p (1 - f(p) p^{-s})^{-1}.$$

PROOF. The absolute convergence of the series

$$(6) \quad \sum_{h=0}^{\infty} f(p^h)p^{-hs}$$

is immediate for $\sigma > \sigma_2$ by comparison with the series

$$\sum_{n=1}^{\infty} |f(n)|n^{-\sigma}.$$

Furthermore, if f is not identically zero, then it is easy to see that $f(1) = 1$, so that the series (6) is now a convergent geometric series with sum $(1 - f(p)p^{-s})^{-1}$. \circ

EXAMPLE. Consider again the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

For every $s \in \mathbb{C}$ satisfying $\sigma > 1$, we have

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

This is called the Euler product of the Riemann zeta function $\zeta(s)$.

PROBLEMS FOR CHAPTER 3

1. Prove that for $\sigma > 1$, we have

$$(i) \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s};$$

$$(ii) \quad \zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s};$$

$$(iii) \quad \frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s};$$

$$(iv) \quad \frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s};$$

$$(v) \quad \frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}.$$

2. Prove that for $\sigma > 2$, we have

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}.$$